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Contribution

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Notations and symbols

$\mathbb{R}$	The set of real numbers
$\Sigma(\mathbb{R})$	The Lebesgue $\sigma-$ algebra
$\mathbb{C}$	The set of complexe numbers
$\mathbb{X}$	Banach space
$\mathbb{X}^*$	The dual space of $\mathbb{X}$
$B(\mathbb{X})$	The closed unit ball of $\mathbb{X}$
$S(\mathbb{X})$	The unit sphere of $\mathbb{X}$
$conv(K)$	Convex hull of the set K.
$\overline{conv}(K)$	Closed convex hull of the set K.
$M(\mathbb{R},\mathbb{X})$	The space of measurable functions defined on $\mathbb{R}$ with values in $\mathbb{X}$
$L^p([0,1],\mathbb{X})$	The space of p-integratable functions defined on $[0,1]$ with values in $\mathbb{X}$
$BUC(\mathbb{R},\mathbb{X})$	The space of bounded and uniformly continuous functions
$\|\cdot\|_\infty$	The sup norm of the function $f:\mathbb{R}\to\mathbb{X}$ defined by : $\|f\|_\infty = \sup_{x\in\mathbb{R}}\|f(x)\|$
ϕ	Orlicz or Young function
$\phi\in\Delta_2$	ϕ verifies the Δ_2- condition
$diamH$	The diameter of the set H
ψ	Complementary function of ϕ
ρ_ϕ	Orlicz modular
L^ϕ	Orlicz space equipped with the Luxemburg norm
L^0_ϕ	Orlicz space equipped with the Orlicz norm
E^ϕ	Subspace of L^ϕ
$\|.\|_\phi$	Luxemburg norm in Orlicz space
$\|.\|^0_\phi$	Orlicz norm in Orlicz space
$Trig(\mathbb{R},\mathbb{C})$	The set of generalized trigonometric polynomials

$AP(\mathbb{R},\mathbb{C})$	The space of Bohr almost periodic functions
$\rho_{B^{\phi}}$	Besicovitch-Orlicz pseudo-modular
$B^{\phi}_{a.p}(\mathbb{R},\mathbb{C})$	Besicovitch-Orlicz space of almost periodic functions $f:\mathbb{R}\to\mathbb{C}$
$B^{\phi}_{a.p}(\mathbb{R})$	Besicovitch-Orlicz space of almost periodic functions $f:\mathbb{R}\to\mathbb{R}$
$\tilde{B}^{\phi}_{a.p}(\mathbb{R})$	Besicovitch-Orlicz space of almost periodic functions $f:\mathbb{R}\to\mathbb{R}$
$\tilde{B}^{\phi}_{a.p}(\mathbb{R},\mathbb{C})$	Besicovitch-Orlicz space of almost periodic functions $f:\mathbb{R}\to\mathbb{C}$
$\|.\|_{B^p}$	Besicovitch norm.
$\|.\|_{B^{\phi}}$	Luxemburg norm in Besicovitch Orlicz space
$\|.\|^{o}_{B^{\phi}}$	Orlicz norm in Besicovitch Orlicz space
$\|.\|_{S^p}$	Stepanov- norm
χ_A	The characteristic function of $A\in\Sigma(\mathbb{R})$
$\|.\|_c$	Strictly convex norm
$extr[K]$	A set of extreme points of K
$Sextr(K)$	A set of strongly extreme points of a set K
$Dent(K)$	A set of denting points of a set K
$L^p_{loc}(\mathbb{R},\mathbb{X})$	The space of locally p-integrable functions
l_∞	The space of numerical bounded sequences with $\|x\|=\sup\lvert x_n\rvert$
l_1	The space of complex-valued sequences such that $\sum_n \lvert x_n\rvert<\infty$
c_0	The space of complex-valued sequences $x=(x_n)$ such that $\lim x_n=0$
$\alpha(K)$	The Kuratowski measure of non compactness of subset K of $\mathbb{X}$
S_ϕ	The set of strictly convex points of ϕ
μ	Lebesgue's measure on $\mathbb{R}$
μ_B	Subadditive measure on $\Sigma(\mathbb{R})$
$\delta_{\mathbb{X}}(.)$	The modulus of convexity of $\mathbb{X}$

Introduction

Extreme points are well known in the geometric theory of Banach spaces, optimization, and convex analysis, and they are without a doubt the most fundamental concepts in the study of the behavior of balls in Banach spaces, as evidenced by a large amount of research in this area (see, e.g., [6, 33, 34, 45, 54, 65, 93, 95]). To justify the importance of this notion, we can cite for example the famous Krein Milman theorem which states that any compact convex set K in a Banach space is the convex hull of its extreme points set.

Extreme points were initially investigated in finite dimensional spaces, then mathematicians have dealt with them in infinite dimensional spaces. These studies have led to useful results in classical Banach spaces. We can cite for example results obtained in Orlicz spaces, see for example [21, 36, 49, 94] and references therein.

Let us recall that, intuitively speaking, a point x in a convex set K is called an extreme point of K if and only if x does not belong to the interior of a line segment contained in K.

The notion of extreme point is also connected with the strict convexity. More precisely, a Banach space $\mathbb{X}$ is said to be strictly convex (or rotund) if every point of $S(\mathbb{X})$ is an extreme point of $B(\mathbb{X})$ i.e., $S(\mathbb{X}) = \mathbf{extr}[B(\mathbb{X})]$.

The criteria for extreme points and strict convexity in classical Orlicz spaces (i.e. Banach spaces of which the L^p spaces are a special case) and Musielak-Orlicz spaces (i.e. spaces which are generalization of Lebesgue spaces with variable exponents $L^{p(x)}$) equipped with the Orlicz norm, the Luxemburg norm, and p-Amemiya norm, has been obtained earlier see for instance [21, 35, 49, 90].

The main topic treated in this thesis is the characterization of extreme points of the unit ball of the Besicovitch-Orlicz space of almost periodic functions $B^{\phi}a.p$.

In order to describe the space $B^{\phi}a.p$, we begin by reviewing the definitions of Orlicz spaces and Bohr almost periodic functions.

Initially, almost periodic functions are defined by the Danish mathematician H.

Bohr [15] in 1923 as a natural generalization of the periodicity. Indeed, a continuous function $f:\mathbb{R}\to\mathbb{C}$ is said to be *Bohr almost periodic*, (we write $f\in AP(\mathbb{R},\mathbb{C})$) if for all $\varepsilon>0$ the set

$$T(f,\varepsilon):=\{\tau\in\mathbb{R},\ \ \|f_\tau-f\|_\infty<\varepsilon\},$$

is relatively dense in $\mathbb{R}$, where f_τ is the translation mapping of f, (see [4], [25], [97].)

Bohr almost periodic functions are connected with several branches of mathematics such as differential equations and optimization. They enjoy important properties, in particular they are bounded, uniformly continuous and can be defined as uniform limits of generalized trigonometric polynomials sequences. More precisely,

$$AP(\mathbb{R},\mathbb{C})=\overline{Trig(\mathbb{R},\mathbb{C})}^{\|\cdot\|_\infty}, \tag{1}$$

where

$$Trig(\mathbb{R},\mathbb{C})=\left\{P(t)=\sum_{k=1}^{n}a_k e^{k\lambda_k t},\ \lambda_k\in\mathbb{R},\ a_k\in\mathbb{C},\ n\in\mathbb{N}\right\}.$$

The equality (1) is the starting point for new generalizations of the concept of almost periodicity. First, it was generalized to include discontinuous functions by V. V. Stepanov in 1926 [92] and subsequently by A. S. Besicovitch [12].

In fact, considering the closure of $Trig(\mathbb{R},\mathbb{C})$ with respect to the following pseudo-norms of L^p type

$$\|f\|_{S^p}=\sup_{x\in\mathbb{R}}\left(\int_x^{x+1}\|f(t)\|^p\,d\mu\right)^{\frac{1}{p}},\ \|f\|_{B^p}=\overline{\lim_{T\to\infty}}\left(\frac{1}{2T}\int_{-T}^{T}\|f(t)\|^p\,d\mu\right)^{\frac{1}{p}}.$$

Stepanov [92] and A.S. Besicovitch [12] have defined, respectively, the spaces called Stepanov ($S^p a.p$) and Besicovitch ($B^p a.p$) spaces of almost periodic functions containing the class of Bohr almost periodic functions.

The theory of Orlicz spaces was appeared in 1932 in the work of W. Orlicz [85], it became usually known with the publishing of a monograph on the subject and its applications by M. A. Krasnoselskii and Ya B. Rutitskii [61]. From the 1980s forward, the geometry of Orlicz spaces, as well as several topological questions, have been extensively researched, the various results obtained are published in monographs by S. Chen [21], J. Musielak [83], M. M. Rao and Z. D. Ren [87], and P. Kosmol et.al [59].

In his well-known paper [47], T.R Hillmann has extended the Bohr almost periodicity within the framework of Orlicz spaces, revealing a new space called Besicovitch-

Orlicz of almost periodic functions. Namely, let ϕ be a Young function. We consider the (pseudo) modular,

$$\rho_{B^\phi}(f) = \overline{\lim_{T\to+\infty}} \frac{1}{2T}\int_{-T}^{T}\phi(|f(t)|)dt, \tag{2}$$

to which we associate the Luxemburg norm

$$\|f\|_{B^\phi} = \inf\{k>0 : \rho_{B^\phi}(\frac{f}{k}) \le 1\}. \tag{3}$$

The modular space $B^\phi(\mathbb{R},\mathbb{C}) = \{f \in M(\mathbb{R},\mathbb{C}),\ \rho_{B^\phi}(\lambda f) < \infty, \text{for some} \lambda > 0\}$ is called Besicovitch-Orlicz space.

The closure of $Trig(\mathbb{R},\mathbb{C})$ with respect to the norm $\|.\|_{B^\phi}$ allows to define the space $B^\phi a.p.(\mathbb{R},\mathbb{C})$ the space of almost periodic functions in the sense of Besicovitch-Orlicz.

The general structure as well as certain topological properties of these spaces are studied in [47]. In the recent years, some geometrical properties of $B^\phi_{a.p.}(\mathbb{R})$, have been considered by M. Morsli and his collaborators in [8, 9, 76, 79, 81].

M. Morsli [79] has discussed the criteria of rotundity of $B^\phi_{a.p.}(\mathbb{R})$ equipped with the Luxemburg norm. He proved that $B^\phi_{a.p.}(\mathbb{R})$ is strictly convex if and only if ϕ is strictly convex and has at most polynomial growth (ϕ satisfies the Δ_2-condition see (2.2.1)).

In [81], M. Morsli and F. Bedouhene have characterized the rotundity of $B^\phi_{a.p.}(\mathbb{R})$, when it is endowed with the Orlicz norm. However, to our knowledge, the criteria for extreme points has not been discussed yet. The main goal of this thesis is to characterize extreme points of the unit ball of $B^\phi_{a.p.}$.

The organization of this thesis is as follows.

In the first chapter, we are concerned with the geometric properties of a general Banach space. First we start with a survey over some of special points of its unit ball : extreme points, strongly extreme points, denting points, exposed points, strongly exposed points. We give characterizations of these points in classical Banach spaces and the relations between them and the strict (uniform) convexity. The results presented are standard and well known. The second section of this chapter summarizes some geometric properties of a Banach space such as B- convexity, uniform non squareness, normal structure, property β. We conclude the chapter with some applications of the geometric properties in fixed point theory as well as the characterizations of the reflexivity which is a topological property. In the second chapter, we first recall definitions and results concerning Orlicz spaces and their geometric. Then we present

Bohr, Besicovitch and Besicovitch-Orlicz almost periodicity. Many results concerning the geometry of the Besicovitch- Orlicz space of almost periodic functions are given in the end of this chapter.

In the third chapter of this thesis, is focused on the extreme points of the unit ball of the Besicovitch-Orlicz space of almost periodic functions.

In the first section, we have established necessary and sufficient conditions for an element of the unit sphere of $B^{\phi}_{a.p}(\mathbb{R},\mathbb{C})$, equipped with the Luxemburg norm to be an extreme point. The conditions are determined by the structure affine intervals and the points of strict convexity of ϕ. The obtained results provides the sufficient conditions for the strict convexity of $B^{\phi}_{a.p}(\mathbb{R},\mathbb{C})$ equipped with the Luxemburg norm, which were presented by M. Morsli in ([79]). In order to demonstrate our main results, we have constructed an example to show that $f\chi_A$ $(A \in \Sigma(\mathbb{R}))$ is not in $B^{\phi}_{a.p}(\mathbb{R},\mathbb{C})$ even when the function f is in $B^{\phi}_{a.p}(\mathbb{R},\mathbb{C})$.

In the second section of this chapter, we have characterized the extreme points of the unit ball of $B^{\phi}_{a.p}(\mathbb{R})$ endowed with the Orlicz norm (equal to Amemiya norm). Before giving our principal result about the extreme point of this space, we have showed several properties of the set $K(f)$ $(f \in B^{\phi}_{a.p}(\mathbb{R}))$, the set of real numbers $k > 0$ for which the infimum is attained in the Amemiya formula.

CHAPTER 1

On the geometry of Banach spaces

When studying the geometry of a Banach space one often seeks to determine the geometry of its unit ball. A common way to distinguish "round" and "flat" parts of the boundary of the unit ball is through extreme and non-extreme points. Among the extreme points, or "round" parts of the boundary, further refinements can be made, for example exposed and strongly exposed points. In this chapter we first review the most important definitions and results about special points of the unit ball of a Banach space (extreme points, strongly extreme points, denting points, exposed points, strongly exposed points), and then we look at how these points are connected to other geometric properties. The last section is devoted to the several conclusions concerning the geometry of Banach spaces applications : characterization of the reflexivity and fixed point theory.

1.1 Special points of the unit sphere of a Banach space

1.1.1 Extreme points

Definition 1.1. *[64]*

A point x in a convex set K is called an extreme point *of K (we write $x \in \mathbf{extr}[K]$) if it* *does not belong to the interior of any open line segment joining tow points of K, i.e,*

$$a_1, a_2 \in K \;\; and \;\; a_1 \neq a_2 \Rightarrow x \notin]a_1, a_2[.$$

In other words, $x \in K$ is said to be an extreme point of K if it can not be written as the arithmetic mean $\frac{1}{2}(y+z)$ of two distinct points $y, z \in K$. Equivalently $x \notin conv(K \backslash \{x\})$.

In [86], M. A. Picardello has used the following Proposition as another definition of extreme point.

Proposition 1.1. *[23] $f \in \mathbf{extr}[B(\mathbb{X})]$ if and only if whenever $g \in \mathbb{X}$ and $\|f \mp g\| \leq 1$ then $g = 0$.*

Proof. *Assume $f \in \mathbf{extr}[B(\mathbb{X})], g \in \mathbb{X}$, and $\|f \pm g\| \leq 1$. Then*

$$f + g \in B(\mathbb{X}), f - g \in B(\mathbb{X}) \text{ and } f = \frac{1}{2}[(f+g) + f - g].$$

Since $f \in \mathbf{extr}[B(\mathbb{X})]$, we get $f + g = f - g$. Which implies that $g = 0$. Now, suppose that $f \in B(\mathbb{X})$ and $x \notin \mathbf{extr}[B(\mathbb{X})]$. Then there exist elements g and h of the unit ball such that $f = \frac{g+h}{2}$ and $h \neq g$. So we can write

$$h = f - \frac{1}{2}(g-h) \text{ and } g = f + \frac{1}{2}(g-h).$$

Hence $\|f \pm \frac{1}{2}(g-h)\| \leq 1$ and $g - h \neq 0$. This is contrary to the hypothesis so it follows that $f \in \mathbf{extr}[B(\mathbb{X})]$.

Convexity and extreme points (see [57])

Rotundity is a basic notion in the investigation of geometry of Banach spaces. Recall that a Banach space $(\mathbb{X}, \|.\|)$ is said to be strictly convex if the mid-point of any line segment joining tow different points on the unit sphere of $\mathbb{X}$ does not lie on it. Namely : a Banach space $\mathbb{X}$ is called strictly convex (or rotund) (write SC) if

$$\forall x, y \in S(\mathbb{X}), \; such \;\; that \;\; x \neq y \;\; then \;\; \|\frac{x+y}{2}\| < 1. \tag{1.1}$$

Example 1.1. $\mathbb{R}^2$ *is strictly convex when it is equipped with Euclidean norm, but it is not strictly convex for the norms $\|.\|_1$, $\|.\|_\infty$*

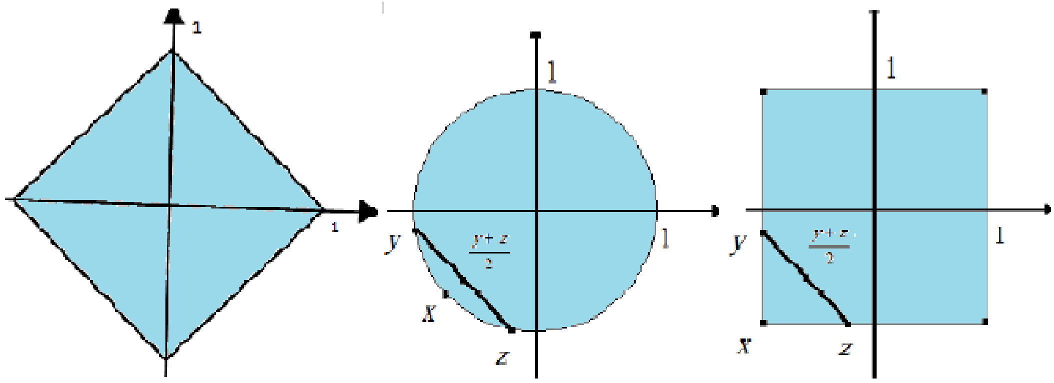

FIGURE 1.1 – Extreme points of $B(\mathbb{R}^2)$

Example 1.2. *The space $C([a,b],\mathbb{R})$ equipped the norm $\|f\| = \sup\{|f(t)| : t \in [a,b]\}$ is not strictly convex. In fact, if we take the functions f, g defined by*

$$f(t) = 1 \quad and \quad g(t) = \frac{t-a}{b-a}, \quad t \in [a,b],$$

we get $\|f\| = \|g\| = 1$, and $f \neq g$ but the mid-point is in the unit sphere, i.e.,

$$\|\frac{f+g}{2}\| = \frac{1}{2}\big(\max_{t\in[a,b]} |1 + \frac{t-a}{b-a}|\big) = 1.$$

Lemma 1.1. *[57] The sum of an arbitrary norm and a strictly convex norm is also a strictly convex norm. More generally, if we define a new norm by the following formula*

$$\|x\|_p = (\|x\|^p + \|x\|_c^p)^{\frac{1}{p}},\ 0 \leq p < \infty,$$

where $\|.\|$ is arbitrary norm and $\|.\|_c$ is a strictly convex norm, then $\|.\|_p$ is also a strictly convex norm.

Theorem 1.1. *(see [21])*
A Banach space. $(\mathbb{X}, \|.\|)$ is strictly convex if and only if any point of $S(\mathbb{X})$ is extreme point of $B(\mathbb{X})$, this means that **extr**$(B(\mathbb{X})) = S(\mathbb{X})$,

Proof. *Suppose that $\mathbb{X}$ is strictly convex. Let $x \in S(\mathbb{X})$ and suppose $x = \frac{y+z}{2}$ with $y, z \in S(\mathbb{X})$ then*

$$\|\frac{y+z}{2}\| = \|x\| = 1,$$

since $\mathbb{X}$ is strictly convex we get $x = y = z$. Thus x is extreme point of $B(\mathbb{X})$.

Now, suppose that any point of $S(\mathbb{X})$ is an extreme point of $B(\mathbb{X})$ then,

$$\forall x,y,z \in S(\mathbb{X}) : x = \frac{y+z}{2} \Rightarrow x = y = z.$$

Let $x, y \in S(\mathbb{X})$ with $x \neq y$ we assume that

$$\|\frac{x+y}{2}\| = 1 \ \text{ which means that } \ (\frac{x+y}{2}) \in S(\mathbb{X}) = \textbf{extr}[B(\mathbb{X})].$$

Thus $x = y$ contradiction with $\|\frac{x+y}{2}\| = 1$. Finally, $\|\frac{x+y}{2}\| < 1$. So $\mathbb{X}$ is strictly convex.

Uniform convexity

The notion of uniformly convex Banach space was introduced by J.A Clarkson in his paper [24] on the theory of integration of functions whose range lies in a Banach space. We observed that in the definition of strict convexity the difference $1 - \|\frac{x+y}{2}\|$ needed not be uniformly bounded from below. This justifies the following definition

Definition 1.2. *[6] A Banach space $(\mathbb{X}, \|.\|)$ is said to be uniformly convex if, for every $\varepsilon > 0$, there is a number $\delta(\varepsilon) > 0$ such that, for all x, y in $\mathbb{X}$,*

$$(\|x\| = \|y\| = 1, \ \ \|x-y\| \geq \varepsilon) \Rightarrow \|\frac{x+y}{2}\| \leq 1 - \delta(\varepsilon).$$

The number

$$\delta_{\mathbb{X}}(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\|, \ \ \|x\| = \|y\| = 1, \ \ \|x-y\| > \varepsilon\},$$

is called the modulus of convexity of $\mathbb{X}$. The space $\mathbb{X}$ is uniformly convex if and only if, $\delta_{\mathbb{X}}(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. Indeed,

1. If $\mathbb{X}$ is uniformly convex, given $\varepsilon > 0$ there exists $\delta > 0$ such that $\delta \leq 1 - \|\frac{x+y}{2}\|$ for every x and y such that $\|x\| = \|y\| = 1$ and $\varepsilon \leq \|x-y\|$. Therefore $\delta_{\mathbb{X}}(\varepsilon) > 0$.
2. For the converse, assume $\delta_{\mathbb{X}}(\varepsilon) > 0$ for every $\varepsilon \in (0,2]$. Fix $\varepsilon \in (0,2]$ and take x, y with $\|x\| = \|y\| = 1$ and $\varepsilon \leq \|x-y\|$, then

$$0 < \delta_{\mathbb{X}}(\varepsilon) \leq 1 - \|\frac{x+y}{2}\|,$$

and therefore $\|\frac{x+y}{2}\| \leq 1 - \delta$ with $\delta = \delta_{\mathbb{X}}(\varepsilon)$ which does not depend on x or y.

Uniform convexity implies strict convexity, and, at least formally, is strictly stronger, since it assumes the differences $1-\|\frac{x+y}{2}\|$ to be uniformly bounded from below, for all x,y in $\mathbb{X}$, $\|x\|=\|y\|=1,\ \|x-y\|\geq\varepsilon$.

Lemma 1.2. *(see [2])*
Let $\mathbb{X}$ be a uniformly convex Banach space. Then we have for any $r>0$ and ε with $r\geq\varepsilon>0$ and elements $x,y\in\mathbb{X}$ with $\|x\|\leq r,\ \|y\|\leq r,\ \|x-y\|\geq\varepsilon$, there exists $\delta=\delta(\frac{\varepsilon}{r})$ such that

$$\|\frac{x+y}{2}\| \quad\leq\quad r[1-\delta(\frac{\varepsilon}{r})]$$

Proof. *Suppose that $\|x\|\leq r,\ \|y\|\leq r,\ \|x-y\|\geq\varepsilon>0$. Then we have*

$$\|\frac{x}{r}\|\leq 1,\ \|\frac{y}{r}\|\leq 1 \text{ and } \|\frac{x}{r}-\frac{y}{r}\|\geq\frac{\varepsilon}{r}>0.$$

By the definition of uniform convexity, there exists $\delta=\delta(\frac{\varepsilon}{r})>0$ such that

$$\|\frac{x+y}{2r}\|\leq(1-\delta),\ \text{which yields } \|\frac{x+y}{2}\|\leq r(1-\delta).$$

Proposition 1.2. *Let H be Hilbert space with $dimH\geq 2$. Then*

$$\delta_H(\varepsilon)=1-\sqrt{1-\frac{1}{4}\varepsilon^2}.$$

In particular.

$$\frac{\varepsilon^2}{8}\leq\delta_H(\varepsilon)\leq\frac{\varepsilon^2}{4}.$$

Proof. *Since H is characterized by the parallelogram identity we have that if $x,y\in H$ satisfy $\|x\|=\|y\|=1$ and $\|x-y\|=\varepsilon$, then*

$$\|\frac{x+y}{2}\|^2=\frac{1}{2}\|x\|^2+\frac{1}{2}\|y\|^2-\frac{1}{4}\|x-y\|^2 \ \text{ and } \ \|\frac{x+y}{2}\|=\sqrt{1-\frac{1}{4}\varepsilon^2}.$$

Consequently,

$$\delta_H(\varepsilon)=1-\sqrt{1-\frac{1}{4}\varepsilon^2}.$$

Moreover, as it is easy to see that

$$1-\sqrt{1-\frac{1}{4}\varepsilon^2}=\frac{\varepsilon^2}{4(1+\sqrt{1-\frac{1}{4}\varepsilon^2})}\geq\frac{\varepsilon^2}{8}\ \ \textit{and}\ \ \frac{\varepsilon^2}{4(1+\sqrt{1-\frac{1}{4}\varepsilon^2})}\leq\frac{\varepsilon^2}{4}.$$

Example 1.3.

1. *The most immediate example of uniformly convex space is Hilbert space. This follows from the Parallelogram identity.*
2. L^p *spaces are uniformly convex for* $1<p<\infty$ *(see [33])*

Remark 1.1. *Any uniformly convex space is strictly convex. The tow concepts are equivalent in finite dimensional space. In the case of infinite dimensional Banach spaces these concepts can be different in the sense that there are strictly convex Banach spaces which are not uniformly convex as it is illustrated by the following examples :*

1. *The space* $C([0,1])$ *with the norm*

$$\|x\|_1=\sup_{t\in[0,1]}|x(t)|+(\int_0^1|x(t)|^2dt)^{\frac{1}{2}}$$

and the space l^1 *equipped with the equivalent norm*

$$\|x\|_0=\sum_{k=1}^{\infty}|x_k|+(\sum_{k=1}^{\infty}|x_k|^2)^{\frac{1}{2}}$$

are strictly convex (see Lemma 1.1) but they are not uniformly convex (see [57]), and so on

2. *The space* l^1 *equipped with the equivalent norm*

$$\|x\|_1=\sum_{k=1}^{\infty}|x_k|+(\sum_{k=1}^{\infty}|\frac{x_k}{k}|^2)^{\frac{1}{2}}$$

is strictly convex (see Lemma 1.1 and theorem 1.1), but it is not uniformly convex. Recall that a Banach space $\mathbb{X}$ *is uniformly convex if and only if for all* $(x_n)_n,(y_n)_n$ *in* $S(\mathbb{X})$ *we have*

$$\lim_{n\to+\infty}\|x_n+y_n\|=2\ \Rightarrow\ \lim_{n\to+\infty}\|x_n-y_n\|=0\ (see\ [16]).$$

Now, take

$$x_n = \frac{n}{n+1}e_n \quad and \quad y_n = x_{n+1}.$$

For $n \in \mathbb{N}$ we have

$$\|x_n\|_1 = \frac{n}{n+1} + \frac{1}{n+1} = 1, \quad and \quad \|y_n\|_1 = 1,$$

and

$$\|x_n - y_n\|_1 = \|x_n + y_n\|_1 = \frac{2n+1}{n+1} + (\frac{1}{(n+1)^2} + \frac{(n+1)^2}{n^2(n+2)^2})^{\frac{1}{2}} \to 2$$

as $n \to \infty$ and it means that $(l^1, \|.\|_1)$ is not uniformly convex.

Extreme points of classical Banach spaces

The purpose of this section is to characterize the extreme points of the unit ball in some well known Banach spaces. We start by recalling that Hilbert spaces are uniformly convex which implies that extreme points of their unit ball are exactly the points of norm one which means $\mathbf{extr}[B(H)] = S(H)$.

Extreme points of sequence spaces c_0 and l_1 (see [23])

Theorem 1.2. *[23]*

1. *$x \in \mathbf{extr}[B(l_1)]$ if and only if $x = \lambda\delta_j$ for some $j \in \mathbb{N}^*$ and some complex number λ with of modulus equal 1. Where $\delta_j = (0,0,...,0,1,0,0,...)$ is the element in l_1 which has a 1 in the j-th position and zeros elsewhere.*
2. *The unit ball of c_0 doesn't contain an extreme point.*

Proof.

1. *First we prove that $x_j = \lambda$, with $|\lambda| = 1$ is an extreme point. Suppose*

$$x = \frac{1}{2}(y+z),$$

where $y,z \in B(l_1)$. Then

$$x_k = \frac{1}{2}(y_k + z_k) = 0, k \in \mathbb{N}^*, k \neq j \text{ and } x_j = \lambda.$$

Since λ is an extreme point of the unit disk in the complex plane we get

$$x_j = y_j = z_j = \lambda.$$

For $k \neq j,\ y_k = z_k = 0$, *because* $\|y\| = \sum_{n\in\mathbb{N}^*} |y_n| \leq 1$ *and* $\|z\| = \sum_{n\in\mathbb{N}^*} |z_n| \leq 1$.
Hence $x = y = z$ *which implies that* $x \in extr[B(l_1)]$.
Now, suppose $\|x\| = 1$ *and* $x \neq \lambda\delta_j$ *for any* $j \in \mathbb{N}^*$, *with* $|\lambda| = 1$.
Let $x_{n_0} = re^{i\theta}$, $n_0 \neq 0$. *Since* $x \neq \lambda\delta_j, \forall j \in \mathbb{N}^*$, *we have* $0 < r < 1$.
Define $y = (y_n), z = (z_n)$ *as follows*

$$y_n = \begin{cases} e^{i\theta} & n = n_0 \\ 0 & n \neq n_0 \end{cases} \quad \text{and} \quad z_n = \begin{cases} 0 & n = n_0 \\ \frac{1}{1-r}x_n & n \neq n_0. \end{cases}$$

Then $\|y\| = 1$ *and* $\|z\| = \sum_{n\in\mathbb{N}^*} |z_n| = \frac{1}{1-r}\left(\sum_{n\in\mathbb{N}^*} |x_n| - |re^{i\theta}|\right) = (1-r)\frac{1}{1-r} = 1$.
Thus $x, y, z \in S(l_1)$ *and* $x = ry + (1-r)z,\ \ 0 < r < 1$, *with* $y \neq z$.
Hence x *is not an extreme point of* $B(l_1)$.

2. *Let* $(x_n) \in B(c_0)$. *Since* $\lim_{n\to+\infty} x_n = 0$ *there exists a positive integer* n_0 *such that* $|x_{n_0}| \leq \frac{1}{2}$. *Let*

$$y_n = \begin{cases} x_n & if \ \ n \neq n_0 \\ x_n + \frac{1}{4} & if \ \ n = n_0 \end{cases} \quad \text{and} \quad z_n = \begin{cases} x_n & if \ \ n \neq n_0 \\ x_n - \frac{1}{4} & if \ \ n = n_0 \end{cases}$$

Then $y_n, z_n \in B(c_0)$, $y_n \neq z_n$ *and* $x_n = \frac{1}{2}(y_n + z_n)$. *Hence* (x_n) *is not an extreme point of* $B(c_0)$.

Extreme points of Lebesgue spaces.

First, note that if $1 < p < +\infty$, *then* $L^p([0,1])$ *is uniformly convex so each point of* $S(L^p([0,1]))$ *is an extreme point of* $B(L^p([0,1]))$ *(see [6, 53]).*

The unit ball in $L^1([0,1])$ *has no extreme point. Indeed, Let* $f : [0,1] \to \mathbb{C}$ *such that*

$$\|f\| = \int_0^1 |f(t)|dt \leq 1$$

Define $g : [0,1] \to \mathbb{R}$ *by*

$$g(t) = \int_0^t |f(x)|dx.$$

g is continuous and $g(0)=0$. Let

$$\psi(t)=g(t)-\frac{1}{2}\int_0^1|f(t)|dt,$$

then by the intermediate value theorem there exists $t_0\in[0,1]$ such that

$$g(t_0)=\frac{1}{2}\int_0^1|f(t)|dt.$$

Define

$$h(t)=2f(t)\chi_{[0,t_0]}(t),\ \ k(t)=2f(t)\chi_{[t_0,1]}(t)$$

we have $k+h=2f$ and $k\neq h$

$$\|h\|=\int_0^1|2f(t)\chi_{[0,t_0]}(t)|dt=2\int_0^{t_0}|f(t)|dt=2g(t_0)=\int_0^1|f(t)|dt\leq 1,$$

and

$$\begin{aligned}\|k\| &= \int_0^1 2|f(t)\chi_{[t_0,1]}(t)|dt=\int_0^1 2|f(t)|dt-\int_0^{t_0}2|f(t)|dt\\ &= 2(\|f\|-g(t_0))=2(\|f\|-\frac{1}{2}\|f\|)\leq 1.\end{aligned}$$

So f is not extreme point of $B(L^1([0,1]))$.

Now, we proceed to the case of the space of all essentially bounded functions $L^\infty([0,1])$. We recall that the norm

$$\|f\|_\infty=\underset{0\leq t\leq 1}{esssup}|f(t)|=\inf\{c\geq 0:\mu\{t\in[0,1]:|f(t)|>c\}=0\}.$$

Theorem 1.3. *[23]*
*$f\in$ **extr**$[L^\infty([0,1]))]$ if and only if $|f(x)|=1$ almost every where on $[0,1]$.*

Proof. *Suppose $|f(x)|<1$ for $x\in P$, where $P=\{x\in[0,1]:|f(x)|>0\}$ and $\mu(P)>0$. Define*

$$\begin{cases} g(x)=f(x)+(1-|f(x)|)\\ h(x)=f(x)-(1-|f(x)|)\end{cases}$$

Since $1-|f(x)|\geq 0$, we get

$$\|g\|_\infty=esssup|f+(1-|f|)|\leq esssup(|f|+1-|f|)=1.$$

Similarly we have $\|h\| \leq 1$. *Therefore* $g,h \in B(L^\infty([0,1]))$.

$$f = \frac{g+h}{2} \ \ \textit{and} \ \ f \neq g$$

Thus $f \notin \textbf{extr}(B(L^\infty([0,1])))$.

Let $|f(x)| = 1$ *a.e. and suppose* $f = \frac{1}{2}(g+h)$ *a.e. where* $g,h \in B(L^\infty([0,1]))$. *Then for almost all* $x \in [0,1]$

$$f(x) = \frac{1}{2}(g(x)+h(x)), \ \ |f(x)| = 1, \ \ |g(x)| \leq 1, \ \ |h(x)| \leq 1$$

Then

$$|f(x)| = |g(x)| = |h(x)| = 1 \ \ \textit{implies} \ \ f = g = h \ \ \textit{a.e.}$$

Extreme points of the space C(K) (see [6])

Extreme points of the unit ball of $C(K)$ *are the continuous functions* f *such that* $|f(x)| = 1$ *for all* $x \in K$.

Indeed, let $f \in B(C(K))$ *and suppose that* $|f(x)| < 1$ *on some nonempty subset of* K. *Define*

$$\begin{cases} g(t) = f(t) + \frac{1}{2}(1-|f(t)|) \\ h(t) = f(t) - \frac{1}{2}(1-|f(t)|) \end{cases}$$

We have $g,h \in C(K)$ *and*

$$|g(t)| \leq |f(t)| + \frac{1}{2}(1-|f(t)|) = \frac{1}{2}(1+|f(t)|) \leq 1, forallt \in K.$$

Similarly $|h(t)| \leq 1$ *for all* $t \in K$.

Therefore $g,h \in B(C(K))$, $f = \frac{1}{2}(g+h)$ *and* $g \neq h$. *This implies that* $f \notin \textbf{extr}[B(C(K))]$ *which prove that the condition is necessary.*
Suppose $|f(t)| = 1$ *for all* $t \in K$ *and let* $g \in C(K)$ *be such that*

$$|f(t) \pm g(t)| \leq 1 \ \ for \ \ all \ \ t \in K,$$

by Proposition 1.1, we have $g \equiv 0$. *Hence* $f \in \textbf{extr}[B(C(K))]$.

1.1.2 Strongly extreme points

The notion of strongly extreme point is more special than that of extreme point. A strongly extreme point is extreme, thus justified the vocabulary of strongly.
Intuitively speaking, a strongly extreme point x of a convex set K is a point of K such that for each real number $r > 0$, segments of length $2r$ and centered x are not uniformly closer to K than some positive number $d(x,r)$. More precisely, we have the following definitions.

Definition 1.3. *[30]*

A point $x \in S(\mathbb{X})$ is said to be strongly extreme point of $B(\mathbb{X})$ if for any sequences $(x_n),(y_n) \subset \mathbb{X}$ with $\|x_n\| \to 1, \|y_n\| \to 1$, and $2x = x_n + y_n$. There implication holds

$$\lim_{n\to+\infty} \|x_n - y_n\| \to 0.$$

We denote by $sextr[B(\mathbb{X})]$ the set of strongly extreme *point of $B(\mathbb{X})$.*

There are equivalent descriptions of strongly extreme points as follows

Definition 1.4. *(see [54])*

1. *A point x in a Banach space $\mathbb{X}$, with $\|x\| = 1$ is a* strong extreme point *of the unit ball of $\mathbb{X}$ if for each $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$\max(\|x+y\|,\|x-y\|) \leq 1+\delta \;\; implies \;\; \|y\| \leq \varepsilon.$$

2. *x is strongly extreme point of $B(\mathbb{X})$ if for any sequences $(x_n),(y_n)$ in $B(\mathbb{X})$ we have*

$$\lim_{n\to\infty} \|x - \frac{y_n + z_n}{2}\| = 0 \Rightarrow \lim_{n\to\infty} \|y_n - z_n\| = 0.$$

   This condition can be replaced by

$$\lim_{n\to\infty} \|x \pm x_n\| = 1 \Rightarrow \lim_{n\to\infty} \|x_n\| = 0.$$

If the set of all strongly extreme points of $B(\mathbb{X})$ is equal to $S(\mathbb{X})$, then $\mathbb{X}$ is said **middle locally uniformly convex.**

Remark 1.2. *(see [45]) Let S be a convex subset of a Banach space $\mathbb{X}$, if S is compact then every extreme point of S is strongly extreme point of S. So in $\mathbb{R}^2$ the two notions (extreme points and strongly extreme points) coincide because $B(\mathbb{R}^2)$ is compact.*

1.1.3 Denting points

The notion of denting point plays an important role because it is connected with the Radon- Nikodým Property (RNP). Namely, H. B. Maynard [70] proved that $\mathbb{X}$ has the RNP if and only if every non-empty bounded closed set K in $\mathbb{X}$ has at least one denting point.

Definition 1.5. *[67] Let K be a bounded closed convex set of a Banach space $\mathbb{X}$. An element x in K is called a denting point of K if*

$$x \notin \overline{conv}(K \backslash B(x,\varepsilon)) \ \ for\ all \ \ \varepsilon > 0,$$

where $B(x,\varepsilon) = \{y : y \in \mathbb{X}, \|y-x\| < \varepsilon\}$.

Definition 1.6. *[14, 67]*

A point $x \in S(\mathbb{X})$ is called a denting point of the unit ball $B(\mathbb{X})$ (we write $x \in Dent(B(\mathbb{X}))$) if we have

$$x \notin \overline{conv}\{B(\mathbb{X}) \backslash [x + \varepsilon B(\mathbb{X})]\}, \ \ for \ \ each \ \ \varepsilon > 0$$

. The following is an equivalent description of denting points (see [45, 54]) :

A point $x \in S(\mathbb{X})$ is a denting point of the unit ball of $\mathbb{X}$ if for each $\varepsilon > 0$ the closed convex hull of the set

$$\{y \in \mathbb{X} : \|y\| \leq 1 \ \ and \ \ \|y-x\| \geq \varepsilon\}$$

dose not contain x.

Remark 1.3. *If x is a* denting point *of $B(\mathbb{X})$ then x is a* strongly extreme *point of $B(\mathbb{X})$. More precisely, we have*

$$Dent[B(\mathbb{X})] \subset sextr[B(\mathbb{X})] \subset \boldsymbol{extr}[B(\mathbb{X})].$$

Example 1.4. *The following figure shows denting points of the unit ball of $\mathbb{R}^2$ equipped with Euclidean norm.*

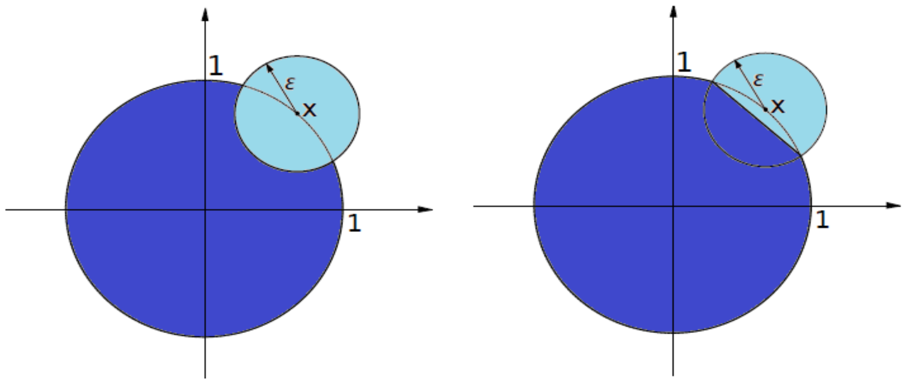

FIGURE 1.2 – Denting points of unit ball of $\mathbb{R}^2$

1.1.4 Exposed points.

$x \in S(\mathbb{X})$ is exposed of $B(\mathbb{X})$ if there exists a functional $L \in \mathbb{X}^*$ witch attains its norm only at x. The functional L is assumed to be of norm 1.

Definition 1.7. *([53]) Let $\mathbb{X}$ be a Banach space and let $A \subseteq \mathbb{X}$. A point $a \in \mathbb{X}$ is called* an exposed point *of A if there exists a bounded linear functional $L \in \mathbb{X}^*$ such that*

$$L(y) < L(a) \quad for\ all \quad y \in A \backslash \{a\}.$$

We call L an exposing functional *(or we say that L expose a).*

The following is an equivalent definition of exposed points.

1. *Given a closed convex set C, a point $x \in C$ is* exposed *if there exists supporting hyperplane H for C such that $H \cap C = \{x\}$ (see [48]).*
2. *A point $x \in S(\mathbb{X})$ is called exposed if there exists a continuous linear functional L on $\mathbb{X}$ such that $L(x) = \|L\| = 1$ and x is the only point in the unit ball that is mapped to 1 (see [10]).*

The set of exposed points of a set A will be denoted $exp(A)$.

Proposition 1.3. *Every exposed point is an extreme point [53].*

Proof. *let $\mathbb{X}$ be a Banach space and let x be an exposed point of $A \subset \mathbb{X}$ with exposing functional L. Suppose there exist $y_1, y_2 \in A$ and $\lambda \in (0,1)$ such that*

$$x = \lambda y_1 + (1 - \lambda) y_2.$$

This implies

$$L(x) = \lambda L(y_1) + (1 - \lambda)L(y_2)$$

However, this means we cannot have both $L(y_1) < L(x)$ and $L(y_2) < L(x)$. This is a contradiction. Therefore x is an extreme point.

The converse of need do not hold in general but we have the following result : If $\mathbf{extr}[B(\mathbb{X})] = S(\mathbb{X})$. Then all points of $S(\mathbb{X})$ are also exposed.

Example 1.5. *[53] In following figure, the point P is exposed but Q isn't because the tangent line at Q intersects the running track in infinitely many points.*

FIGURE 1.3 – Extreme non exposed point

Example 1.6. *[53] For $1 < p < \infty$ the exposed points of the unit ball in $L^p(\mathbb{X})$ are the functions of norm one.*

Example 1.7. *([68]) Consider the set $K = conv(\{(1,1)\} \cup B(\mathbb{R}^2)$ when $\mathbb{R}^2$ is equipped with Euclidean norm. Then $x = (1,0)$ is a denting point of K which is not exposed.*

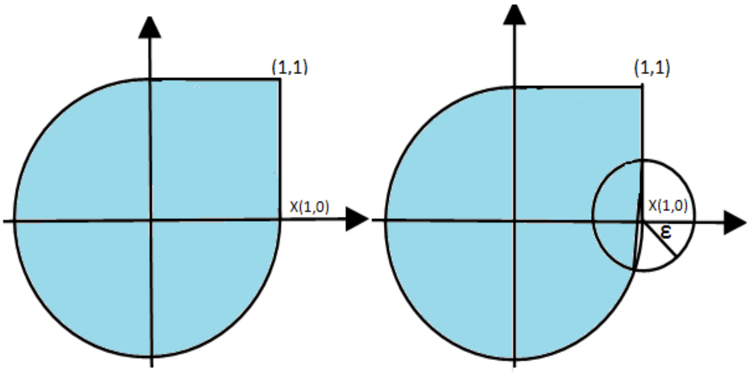

FIGURE 1.4 – The set K ,$\overline{conv}$(K$\backslash B(x,\varepsilon)$)

1.2 Other geometric properties of a Banach space

1.2.1 B-convexity and uniform non-squareness

The B-convexity was introduced by A. Beck [7] as a characterization of Banach spaces $\mathbb{X}$ for which the Strong Law of Large Numbers is valid for $\mathbb{X}$-valued random variables. A general study of B-convexity is given in [7] and [51].

A Banach space $\mathbb{X}$ is called uniformly non-l_n^1 ((n, ε)-convex) according to the terminology used by D.P.Giesty [39] if there is a number $\varepsilon > 0$ such that for each n elements $x_1, x_2, \ldots, x_n$ of $S(\mathbb{X})$ it is true that

$$\frac{1}{n}\|x_1 \pm x_2 \pm \ldots \pm x_n\| \leq 1 - \varepsilon,$$

for some choice of signs.
A Banach space $\mathbb{X}$ is called B-convex if it is uniformly non-l_n^1 for some $n \geq 2$ [51, 63].

Geometrically, a uniformly non-l_n^1 space is one which does not have dimensional subspaces whose norms are arbitrarily good approximations of the l_1 norm (see [39, 51, 63]). Note that each uniformly non-l_n^1 Banach space is uniformly non-l_{n+1}^1 (see [77]).

Remark 1.4. *When $\mathbb{X}$ is finite dimension, the uniformly non-l_n^1 of $\mathbb{X}$ is said non-l_n^1. It characterized by the number of extreme points of its unit ball, more precisely $\mathbb{X}$ is non-l_n^1 if and only if*

$$Card\,(\textbf{extr}[B(\mathbb{X})]) \geq 2n + 2.$$

Example 1.8.

1. *The space $l_n^1 = \{(x_1,\ x_2, \ldots,\ x_n),\ \|x\|_1 = \sum_{i=1}^{n} |x_i| < \infty\}$ is not uniformly non-l_n^1, because*

$$Card(\textbf{extr}(B(l_n^1))) = Card(\{\mp e_i, i = 1, 2, \ldots, n\}) = 2n$$

2. *For the space $l_n^\infty = \{(x_1, x_2, \ldots, x_n), \|x\|_\infty = \max_{i=1,n} |x_1| < \infty\}$, we can see that we have*

$$Card(\textbf{extr}[B(l_n^\infty)]) = 2^n,$$

we will notice that if $n = 2$, then $2^n < 2(n+1)$ and so l_2^∞ is not uniformly non-l_2^1. However l_n^∞ is uniformly non-l_n^1 for every $n \geq 3$.

Uniform non-squareness

Uniform non-l_2^1 Banach spaces are called uniform non-square (see [6,51]), uniform non-squareness of Banach spaces has been defined by R.C. James [51] as the geometric property which implies super-reflexivity.

A Banach space $(\mathbb{X}, \|.\|)$ is non-square if for any x and y from $S(\mathbb{X})$ we have

$$\min(\frac{\|x+y\|}{2}, \frac{\|x-y\|}{2}) < 1,$$

and it is uniformly non-square if there exists $\delta \in (0,1)$ such that for any $x, y \in S(\mathbb{X})$, we have

$$\frac{\|x+y\|}{2} \leq 1-\delta \;\; or \;\; \frac{\|x-y\|}{2} \leq 1-\delta.$$

This condition is of the same kind as uniform convexity, but involves only points which are large distance from each other. In the definition of uniform convexity we know that $\|\frac{x+y}{2}\| \leq 1-\delta$ if $\|x-y\| > \varepsilon$, here we have the same condition if $\varepsilon = 2(1-\delta)$. Note that each uniformly convex Banach space $\mathbb{X}$ is uniform-non square, but the inverse statement is not true : there exist uniformly non-square Banach spaces that are not rotund. Example $\mathbb{R}^2$ equipped with the norm $\|(x_1, x_2) = \| \max(2|x_1|, |x_1| + |x_2|)$ is not rotund (see the following figure). But it is uniformly non square. Namely

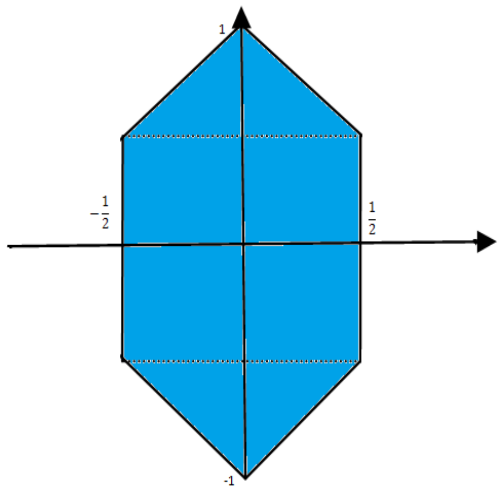

FIGURE 1.5 – The unit ball $B(\mathbb{R}^2)$ equipped with $\max(2|x_1|, |x_1 + x_2|)$

$$\begin{aligned} B(\mathbb{R}^2) &= \{x \in \mathbb{R}^2 : \|x\| \leq 1\} \\ &= \{x \in \mathbb{R}^2 : |x_1| \leq \frac{1}{2} \;\; et \;\; |x_2| \leq 1\} \cap \{x \in \mathbb{R}^2 : |x_1| + |x_2| \leq 1\}. \end{aligned}$$

Since $B(\mathbb{R}^2)$ is compact, and $\min(\|x+y\|, \|x-y\|) < 2$ for any $x, y \in B(\mathbb{R}^2)$, we have

$$\sup_{x,y \in B(\mathbb{R}^2)} min(\|\frac{x+y}{2}\|, \|\frac{x-y}{2}\|) < 1.$$

1.2.2 Normal structure

The concept of normal structure was introduced in 1948 by M.S. Brodskii et al. [18] to study fixed points of certain application and it is a property shared by all uniformly convex spaces. In 1965, W.A. Kirk [56] observed that normal structure implies that a closed bounded convex subset C of $\mathbb{X}$ has the fixed point property if $\mathbb{X}$ is a reflexive Banach space.

Definition 1.8. *[89]*
A Banach space $\mathbb{X}$ is said to have normal structure *if every bounded and convex subset of $\mathbb{X}$ has normal structure.*

A nonempty bounded, convex subset K of a Banach space $\mathbb{X}$ is said to have normal structure if every convex subset H of K, that contains more then one point, there is a point $x_0 \in H$ such that

$$r_H(x_0) = \sup\{\|x_0 - y\| : y \in H\} < diam(H),$$

where $diam(H) = \sup\{\|x-y\| : x, y \in H\}$ denotes the diameter of H (see the following figure).

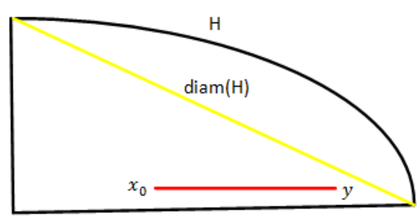

FIGURE 1.6 – The set H has the normal structure.

The following notions play an important role in the study of normal structure. Let C be a nonempty bounded subset of a Banach space $\mathbb{X}$. Then a point $x_0 \in C$ is said to be *a diametral point* of C if

$$\sup\{\|x_0 - x\| : \ x \in C\} = diam(C).$$

A bounded sequence $(x_n)_n$ in a Banach space is said to be a diametral sequence if

$$\lim_{n\to\infty} d(x_{n+1}, conv(\{x_1, x_2, ..., x_n\})) = diam(\{x_n\}),$$

where $d(x,A) = \inf_{y\in A} d(x,y)$.

Theorem 1.4. *[2]*
Every compact convex subset C of a Banach space $\mathbb{X}$ has normal structure.

Proof. *Suppose, for contradiction, that C dose not have normal structure. Let D be a convex subset of C that has at least two points. Because C dose not have normal structure, all point of D are diametral. Now we construct a sequence $(x_i)_{i=1}^{\infty}$ in D such that*

$$\|x_i - x_j\| \quad = \quad diam(D) \; for \;\; all \; i, j \in \mathbb{N}, i \neq j.$$

For this, let x_1 be an arbitrary point in D, then there exists a point $x_2 \in D$ such that $diam(D) = \|x_1 - x_2\|$. Because D is convex, there exists a point $\frac{x_1+x_2}{2} \in D$. Next we choose a point $x_3 \in D$ such that

$$diam(D) = \|x_3 - \frac{x_1 + x_2}{2}\|.$$

Proceeding in the same manner, we obtain a sequence (x_n) in D such that

$$diam(D) = \|x_{n+1} - \frac{x_1 + x_2 + ... + x_n}{n}\|, \; n \geq 2.$$

Because

$$\begin{aligned} diam(D) &= \|x_{n+1} - \frac{x_1 + x_2 + ... + x_n}{n}\| \\ &= \|\frac{(x_{n+1} - x_1) + (x_{n+1} - x_2) + ... + (x_{n+1} - x_n)}{n}\| \\ &\leq \frac{1}{n}(\|x_{n+1} - x_1\| + \|x_{n+1} - x_2\| + ... + \|x_{n+1} - x_n\|) = diam(D) \end{aligned}$$

it follows that $diam(D) = \|x_{n+1} - x_i\|$, $1 \leq i \leq n$. *This implies that the sequence* (x_n) *has no convergent subsequences. This contradicts the compactness of* C.

Corollary 1.1. *([2])*
Every finite dimensional Banach space has normal structure.

Theorem 1.5. *Every closed convex bounded subset* C *of uniformly convex Banach space* $\mathbb{X}$ *has normal structure.*

Proof. *Let* D *be a closed convex subset of* C *with* $diam(D) = d > 0$. *Let* x_1 *be an arbitrary point in* D.
Choose a point $x_2 \in D$ *such that* $\|x_1 - x_2\| \geq \frac{d}{2}$. *Because* D *is convex,* $\frac{x_1+x_2}{2} \in D$. *Set* $x_0 = \frac{x_1+x_2}{2}$. *By the uniform convexity,*

$$\|u\| \leq r, \|v\| \leq r \ \text{ and } \ \|u - v\| \geq \varepsilon > 0 \Rightarrow \|\frac{u+v}{2}\| \leq (1 - \delta_{\mathbb{X}}(\frac{\varepsilon}{r}))r.$$

Hence for $x \in D$ *we have*

$$\begin{aligned} \|x - x_0\| &= \|x - \frac{x_1 + x_2}{2}\| = \|\frac{(x - x_1) + (x - x_2)}{2}\| \\ &\leq d(1 - \delta_{\mathbb{X}}(\frac{d}{2d})) = d(1 - \delta_{\mathbb{X}}(\frac{1}{2})) < d \ \ (as\ \delta_{\mathbb{X}}(\frac{1}{2}) > 0). \end{aligned}$$

Then

$$\|x - x_0\| \leq d(1 - \delta_{\mathbb{X}}(\frac{1}{2})) < d \tag{1.2}$$

Consequently,

$$\sup \|x - x_0\| : x \in D\} < diam(D).$$

Example 1.9. *The space* $C([0,1])$ *of continuous real-valued functions with sup norm doesn't have normal structure.*
To see it, consider the subset C *of* $C([0,1])$ *defined by*

$$C = \{f \in C([0,1]) : 0 = f(0) \leq f(t) \leq f(1) = 1, \ \ t \in [0,1]\}.$$

Let $f_1, f_2 \in C$, $\lambda \in (0,1)$ *and* $f = \lambda f_1 + (1-\lambda) f_2$. *Then* $f(0) = 0, f(1) = 1$ *and* $0 \leq f(t) \leq 1$ *for all* $t \in [0,1]$. *Hence* C *is convex. Thus,* C *is a closed convex bounded subset of* $C([0,1]$ *with* $diam(C) = \sup\{\|f - g\| : f, g \in C\} = 1$. *Then each point of* C *is a diametral point. In* *fact, for* $f_0 \in C$

$$\sup\{\|f_0 - f\| : f \in C\} = 1 = diam(C).$$

Therefore, C dose'nt have normal structure.

Proposition 1.4. *[2]*
A convex bounded subset C of a Banach space $\mathbb{X}$ *has normal structure if and only if it dose not contain a diametral sequence.*

Example 1.10. *In the space* l_1*, the basis vectors* $\{e_n\}$ *form a diametral sequence. Hence* l_1 *dose not have normal structure.*

Uniform normal structure

Definition 1.9. *[89]*
$\mathbb{X}$ *is said to have* uniform normal structure *if there exists* $0 < c < 1$ *such that for any subset K as above, there exists* $x_0 \in K$ *such that*

$$\sup\{\|x_0 - y\|, y \in K\} < c.diam(K).$$

Proposition 1.5. *The following assertions hold :*

1. *Let* $\mathbb{X}$ *be a Banach space with* $\delta_{\mathbb{X}}(\frac{3}{2}) > \frac{1}{4}$*, Then* $\mathbb{X}$ *has uniform normal structure [37].*
2. *If* $\varepsilon_0(\mathbb{X}) < 1$*, then* $\mathbb{X}$ *has normal structure (where* $\varepsilon_0(\mathbb{X}) = \sup\{\varepsilon \in (0,2], \delta_{\mathbb{X}}(\varepsilon) = 0\}$ *the coefficient of convexity of* $\mathbb{X}$*)(see([40])).*

Theorem 1.6. *Every uniformly convex Banach space* $\mathbb{X}$ *has uniformly normal structure.*

Proof. *For a closed convex bounded subset C of* $\mathbb{X}$ *with* $d = diam(C) > 0$ *from (1.2), there exists a point* $x_0 \in C$ *such that*

$$\|x - x_0\| \leq (1 - \delta_{\mathbb{X}}(\frac{1}{2}))d.$$

This implies that

$$\sup\{\|x - x_0\| : x \in C\} \leq \alpha diam(C),$$

where $\alpha = 1 - \delta_{\mathbb{X}}(\frac{1}{2})$*. Therefore,* $\mathbb{X}$ *has uniformly normal structure.*

1.2.3 Property β

In 1987, S. Rolewicz [88] introduced the property called by him property β. It is intermediate between convexity and nearly uniform convexity, and defined by the drop property and the Kuratowski measure.

Recall that

1. $\mathbb{X}$ is said to be **nearly uniformly convex** (we write **NUC**) if for every $\varepsilon > 0$ there exists $\delta \in]0,1[$ such that for every sequence $(x_n) \subset B(\mathbb{X})$ with $sep(x_n) = \inf\{\|x_n - x_m\| : n \neq m\} \geq \varepsilon$, we have

$$conv((x_n)) \cap (1-\delta)B(\mathbb{X}) \neq \emptyset.$$

2. $\mathbb{X}$ has the **drop property** (we write **(D)**) if for every closed set C disjoint with $B(\mathbb{X})$ there exists an element $x \in C$ such that

$$D(x, B(\mathbb{X})) \cap B(\mathbb{X}) = \{x\}.$$

$D(x, B(\mathbb{X})) = conv(\{x\} \cup B(\mathbb{X}))$ is the **drop** determined by x, $(x \notin B(\mathbb{X}))$.

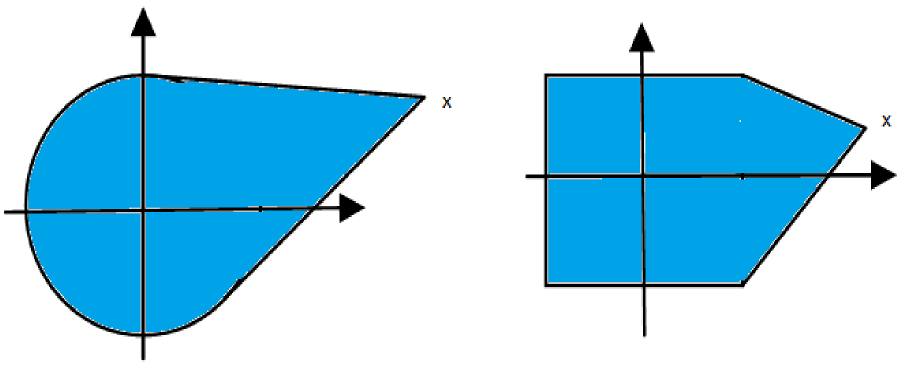

FIGURE 1.7 – **Drop** $D(x, B(\mathbb{R}^2))$ determined by x.

Definition 1.10. *([28, 31, 72]) A Banach space $\mathbb{X}$ is said to have property (β) if for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\alpha(D(x, B(\mathbb{X})) \backslash B(\mathbb{X})) < \varepsilon,$$

whenever $1 < \|x\| < 1 + \delta$.

*$\alpha(D(x,B(\mathbb{X}))$ is the **Kuratowski measure of non compactness** of $D(x,B(\mathbb{X}))$. It is the infimum of such $\varepsilon > 0$ for which there is a covering of $D(x,B(\mathbb{X})$ of a finite number of sets of diameter less then ε.*

Definition 1.11. *[8, 28, 31] A Banach space $\mathbb{X}$ has property (β) if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each element $x \in B(\mathbb{X})$ and each sequence (x_n) in $B(\mathbb{X})$ with $sep(x_n) \geq \varepsilon$ there is an index k for which*

$$\|\frac{x+x_k}{2}\| \leq 1-\delta.$$

Summarizing the above discussion we have

$$(\textbf{UNC}) \Rightarrow (property$$

$\beta)\Rightarrow$ (**NUC**) $\Rightarrow$ **(D)** $\Rightarrow$ **(Reflexive)**.

1.2.4 P-convexity

A measure of the "massiveness" of the unit ball of a Banach space $\mathbb{X}$ is introduced in terms of an efficiency of the tightest packing of balls of equal size in the unit ball. Recall that family of balls $\{B(x_i,r)\}_{i\in I}$ of radius r and center x_i can be packed in the unit ball $B(\mathbb{X})$ if

$$B(x_i,r) \subset B(\mathbb{X}),\ \forall i \in I,\ \text{and } \text{int}B(x_i,r) \cap intB(x_j,r) = \emptyset,\ \forall i \neq j,$$

where $intB(x_i,r)$ is the interior of $B(x_i,r)$.

In 1970 C A. Kottman [63] introduced the P-convexity property, as an evaluation of the efficiency of the tightest packing of balls of equal size in the unit ball of this space. Namely, A Banach space $(\mathbb{X},\|.\|)$ is said to be $P-$convex if

$$P(n,\mathbb{X}) = Sup\{r : \text{there exist } n \text{ disjoint balls of radius } r \text{ in } B(\mathbb{X})\} < \frac{1}{2},$$

for some natural number n

This definition is equivalent to the following

Definition 1.12. *A Banach space $\mathbb{X}$ is $P-$convex if there exists $\varepsilon > 0$ and $n \in \mathbb{N}$ such that* *for any* $x_1, x_2, ..., x_n \in S(\mathbb{X})$

$$min\left\{\left\|x_i - x_j\right\|, i, j \leq n,\ i \neq j\right\} \leq 2 - \varepsilon$$

In the following, we summarize some properties of the $P-$convexity property.

1. $P-$convexity implies $B-$convexity
2. Every uniformly convex space is $P-$convex.
3. Every $P-$convex space is super-reflexive.

1.3 Some applications

In this section we discuss some applications of geometric properties in the fixed point theory and in the characterization of the reflexivity of Banach spaces.

1.3.1 Characterization of the reflexivity

First, recall that a Banach space $\mathbb{X}$ is reflexive if the canonical injection from $\mathbb{X}$ into $\mathbb{X}^{**}$ is surjective. Reflexivity is a topological property i.e., a reflexive space remains reflexive for an equivalent norm. This property plays an important role in the theory of Banach spaces in particular "$B(\mathbb{X})$ is compact in the weak topology $\sigma(\mathbb{X}, \mathbb{X}^*)$ if and only if $\mathbb{X}$ is reflexive", and every reflexive space $\mathbb{X}$ has the Radon-Nikodym Property.

Remember that Hilbert spaces as finite-dimensional spaces are reflexive, and if $1 < p < \infty$ the Lebesgue spaces L^p and l^p are reflexive. But L^1, L^∞, l^∞ are not reflexive.

In practise, the above definition of reflexivity is difficult, so it was necessary to introduce other, simpler properties (e.g., related to the norm) to characterize reflexivity.

The first contribution in this sense is due to D. Milman in 1938 [71] according to which "any uniformly convex space is reflexive". This result is paradoxical : a topological property implied by a metric property.

We note that the same result holds if $\mathbb{X}$ is not uniformly convex but can be given a new norm defining the same topology under which the space is uniformly convex.

Millman's result is restrictive since "uniform convexity" is a strong geometric property. As a result, other geometric characterizations of the reflexivity are given :

F. Smulian [91] has characterized a reflexive Banach space as follows : $\mathbb{X}$ is reflexive if and only if every decreasing sequence of nonempty bounded closed convex subsets of $\mathbb{X}$ has a nonempty intersection.

In 1964, James [52] gave his following famous intrinsic geometric characterization of reflexivity

Theorem 1.7. *A Banach space $\mathbb{X}$ is reflexive if and only if there is a $\theta \in]0,1[$ such that if $(x_n)_{n\geq 1}$ is a sequence of elements of the unit sphere of $\mathbb{X}$, with $\|u\| > 0$ for all $u \in conv\{x_1,x_2,...\}$, then there are $n_O \in \mathbb{N}, u \in conv\{x_1,x_2,.....,x_{n_o}\}$ and $v \in conv\{x_{n_O+1},x_{n_O+1}.,..\}$ such that $\|u-v\| < 0$.*

Looking for some weaker geometric condition implying reflexivity, Giesy [39], James [51] raised the question whether Banach spaces which are uniformly non$-l_n^1$ with some positive integer $n \geq 2$ are reflexive. James [51] settled the question affirmatively for $n = 2$, in other words he proved that uniformly non-square Banach spaces are reflexive, and gave a partial result for $n = 3$.

Different uniform geometrical properties have been defined between the uniform convexity and the reflexivity of Banach spaces. Relationships between theme and reflexivity have been developed by many authors. It was proved in [2] that a Banach space with a uniformly normal structure is reflexive. C A. Kottman [63] has showed that $P-$convex Banach spaces are reflexive. In the same purpose, S. Rolewicz [88] proved that if a Banach space has the property-β then it is reflexive.

1.3.2 Fixed point property

The aim of this section is to present some results concerning geometric properties related to the metric fixed point theory. This theory has been a large subject of mathematical research for decades due to its many diverse applications, example in differential and integral equations.

The most known and important result in metric fixed point theory is the Banach fixed point theorem also called the Contractive Mapping Principle, given by S. Banach in 1922, which assures that every contraction from a complete metric space into itself has a unique fixed point.

We recall that a mapping T from a Banach space $(\mathbb{X}, \|.\|)$ into it self is said to be a contraction if there exist $0 < k < 1$ such that $\|T(x) - T(y)\| \leq k\|x-y\|$, for all $x,y \in \mathbb{X}$. The Banach theorem does not hold if $k = 1$ that is $\|T(x) - T(y)\| \leq \|x-y\|$ for every distinct points $x,y \in \mathbb{X}$. Such $1-$Lipschitzian mappings are called nonexpansive. The problem of the existence of a fixed point for such functions was neglected for several years. However, these mappings have received a lot of attention in recent works in this field.

For simplicity, it is usual to say that a Banach space $(\mathbb{X}, \|.\|)$ has the fixed point property (FPP) if every nonexpansive self mapping of every nonempty, closed convex bounded subset C of $\mathbb{X}$ has a fixed point. When the same holds for every weakly compact convex subset of $\mathbb{X}$ we say that $\mathbb{X}$ has the weak fixed point property for nonexpansive mappings (WFPP in short). For reflexive Banach spaces both properties are obviously the same.

The FPP is strongly influenced by the geometric properties of the norm of the space $\mathbb{X}$. In 1965, Browder [19] proved that every nonexpansive mapping T from a convex bounded closed subset C of a Hilbert space $\mathbb{X}$ into C has a fixed point, and Browder [20] and Gôhde [42] proved independently that the previous result could be improved assuming the weaker condition that $\mathbb{X}$ is uniformly convex. More precisely, every uniformly convex Banach space enjoys fixed point property (FPP).

In the same year, W.A. Kirk [56] gave a more general sufficient condition for (FPP). Namely, every reflexive Banach space with normal structure has (FPP). In other words, it is sufficient to assume that the set is weakly compact and the space has the normal structure to have the fixed point property. Kirk's use of normal structure in his result has led to further works which tries to find more general conditions on the Banach space $\mathbb{X}$ and the subset C which still assures the existence of fixed point.

As mentioned in the section on normal structure, compact convex subset of a Banach space $\mathbb{X}$ always has normal structure. The problem whether every weakly compact convex subset of $\mathbb{X}$ has normal structure was answered negatively by Alspach [3]. He showed that there is a weakly compact convex subset C of $L^1([0,1])$ which does not have the fixed point property. In particular, C cannot have normal structure.

In their paper [41], A. A. Gillespie and B. B. Williams replaced uniformly convex (or reflexive and normal structure) as required by Browder and Kirk, by uniformly normal structure to obtain a fixed point theorem for non-expansive self mappings.

Building from the initial results of Browder, Gôhde and Kirk we now have a rich, though still far from complete, theory of nonexpansive in Banach spaces.To our knowledge, the problème wether" every reflexive Banach space have (FPP)" remains unsolved, and it may be the oldest and most difficult in this theory.

We shall now turn to other geometric properties that implies FPP. It is well known that uniform non-squareness implies reflexivity (see [52]), but the new interest on this property was motivated by its connection with the fixed point property. In 2006, J.García-Falset et al [38] proved that uniformly non-square Banach spaces have the fixed point property.

The importance of the property (β) is related to the following assertions :
If $\mathbb{X}$ has the property (β) then it is reflexive and, $\mathbb{X}$ and $\mathbb{X}^*$ have the fixed point property.The following figure gives link between the different geometric properties, the reflexivity and FPP of a Banach space.

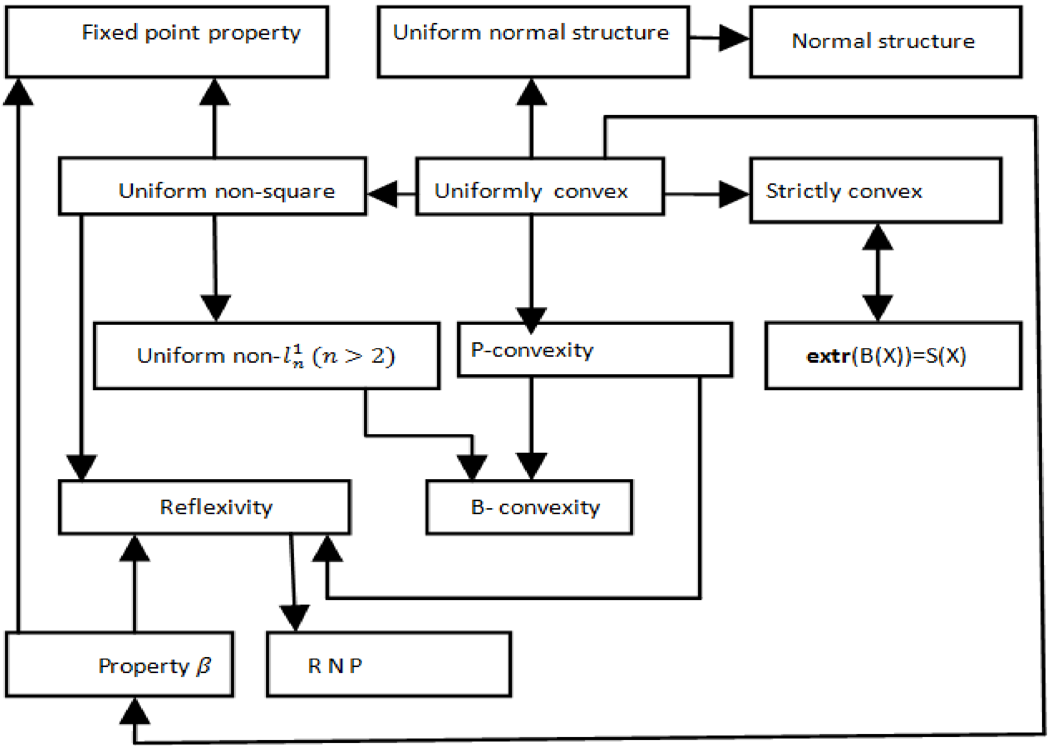

FIGURE 1.8 – Link between the different geometric properties, the reflexivity and (FPP)

CHAPTER 2

Besicovitch-Orlicz spaces of almost periodic functions

2.1 Introduction

The theory of almost periodic functions was initiated by H. Bohr [15], and then developed by A.S. Besicovitch [12, 13] within the framework of Lebesgue spaces L^p.

T.R Hillmann [47], has widened this theory to Orlicz spaces thus emphasis three new classes of spaces : Stepanov-Orlicz $S^{\phi}_{a.p}$, Weyl-Orlicz $W^{\phi}_{a.p}$ and Besicovitch-Orlicz $B^{\phi}_{a.p}$ of almost periodic functions.

We start this chapter with recalling necessary definitions and results of Young functions and modular spaces. Then we summarize the known results about Orlicz spaces, in particular those about their extreme points. The third and the forth sections are devoted respectively to Besicovitch-Orlicz spaces and Bohr almost periodicity.

In the last section we introduce the Besicovitch-Orlicz almost periodic functions and we synthesize some results obtained in the study of geometric properties of $B^{\phi}_{a.p}$.

2.2 Orlicz spaces

The theory of Orlicz spaces has been developed and advanced in a variety of directions due to its significant theoretical properties and imposed applications in various domains of mathematical analysis including nonlinear analysis see for examples [21, 62, 83, 87].

Among the reasons for the development of this theory the many applications to differential and integral equations with kernels of non power types. Krasnoselskii and Rutickii [61] have shown that the use of Orlicz spaces is essential to obtain existence theorems for solutions of non linear integral equations of exponential type. Similarly, the study of nonlinear partial differential in the case where the coefficients are of exponential type has lead mathematicians to look for solutions in generalized sobolev spaces constructed using Orlicz spaces. It's interesting to note that the fact that Orlicz spaces are Banach and modular spaces adds to their richness.

In order to define Orlicz spaces L^{ϕ}, the famous Polish mathematician W. Orlicz in 1930 [85] extended the Lebesgue spaces L^p. Indeed, the power function $t \mapsto |t|^p$, $p \geq 1$ entering the definition of L^p is replaced by a more general convex function ϕ called a Young function and sometimes Orlicz function. These functions have properties similar to those of the powers functions. For a detailed account of these spaces we refer to [21, 61, 62, 87] and references therein.

Before presenting these spaces and their basic properties. We'll begin with a review of modular spaces.

The theory of modular spaces englobe a large class of functional spaces, including Orlicz spaces [85], generalized Lebesgue spaces [32], Musielak-Orlicz spaces [83], Besicovitch spaces [11], Orlicz Lorentz spaces and many others. This theory was initiated by Nakano [84] in 1950 and redefined and generalized by Musielak and Orlicz [87] in 1959.

Thanks to the theory of modular spaces, new generalized Orlicz spaces have been introduced : Stepanov-Orlicz, Weyl-Orlicz, and Besicovitch-Orlicz spaces (see [47]).

For more details about modular spaces, we refer to W. Kozlowski's monograph "Modular Function Spaces" [60], the book of M. Khamsi and Kozlowski [55] and Musielak's book [82].
Here is a brief summary of basic concepts in modular spaces.

Definition 2.1. *A functional $\rho : \mathbb{X} \to [0, +\infty]$ is a pseudomodular if it satisfies*

(i) $\rho(0) = 0$,

(ii) $\rho(x) = \rho(-x), \;\; \forall x \in \mathbb{X}$,

(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y), \;\; \forall \alpha, \beta \geq 0, \alpha + \beta = 1, \; \forall x, y \in \mathbb{X}$.

ρ is a modular when, instead of (i), we have

$$\rho(x) = 0 \text{ if and only if } x = 0.$$

If the condition (iii) is replaced by

$$\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y), \ \ \forall \alpha, \beta \geq 0, \alpha + \beta = 1, \ \forall x, y \in \mathbb{X},$$

the pseudomodular ρ is called convex.

The linear space $X_\rho = \{x \in X : \lim\limits_{\alpha \to 0} \rho(\alpha x) = 0\}$ associated to the modular ρ is called a modular space.

Norms in modular spaces

When ρ is a convex modular the modular X_ρ coincide with

$$\mathbb{X}^*_\rho = \{x \in \mathbb{X}, \ \exists \lambda = \lambda(x) > 0, \text{ s.t } \rho(\lambda x) < +\infty\},$$

and the formula

$$\|x\|_\rho = \inf\left\{k > 0, \ \rho\left(\frac{x}{k}\right) \leq 1\right\}$$

defines a norm in the modular space X_ρ (see [83, 87]), which is frequently called the Luxemburg norm.

Note that if ρ is not convex, then $\|.\|_\rho$ is not a norm.

Another norm called "Amemiya norm " is defined in X_ρ by

$$\|f\|^A_\rho = \inf_{\lambda > 0} \frac{1}{\lambda}\{1 + (\lambda f)\}.$$

In recent years, there was an increasing interest in the study of the geometry of modular spaces and their fixed points see for example [55].

2.2.1 Young functions

First, we recall that a real function defined on an interval I of $\mathbb{R}$ is called convex if its graph is "turned upwards". In other words, for all points a and b of its graph, the segment $[a, b]$ is entirely located above the graph. Namely : a function ϕ defined on an interval I of $\mathbb{R}$ is convex if

$$\forall x,\, y \in I,\, \forall \lambda \in [0,\, 1] : \phi(\lambda x + (1-\lambda)y) \leq \lambda \phi(x) + (1-\lambda)\phi(y). \tag{2.1}$$

In this section we present a collection of important notions and facts about Young (Orlicz) functions that we will need for the studying of Orlicz spaces and almost periodic functions of Orlicz type. At first we will investigate the properties of these functions and their conjugates.

A function $\phi : \mathbb{R} \to \mathbb{R}^+$ is said to be a Young function if it is even, convex, and satisfies $\phi(0) = 0, \phi(u) > 0$ if $u \neq 0$ moreover $\lim\limits_{u \to \infty} \phi(u) = +\infty$.
If we have also $\lim\limits_{x \to 0} \frac{\phi(x)}{x} = 0$ and $\lim\limits_{x \to +\infty} \frac{\phi(x)}{x} = +\infty$, the function ϕ is called Orlicz function (sometimes $N-$function). Note that a Young function ϕ is strictly increasing and continuous on $[0, +\infty[$.
The following functions

$$\phi_1(x) = \exp(|x|) - 1,\ \phi_p(x) = \frac{|x|^p}{p}, 1 \leq p < +\infty,$$

and

$$\phi_{\infty,1}(x) = \begin{cases} 0 & \text{if } x \in [-1,\, 1] \\ |x| - 1 & \text{or else.} \end{cases}$$

are Young functions and ϕ_p is an $N-$fonction.

For every Young function ϕ we define the complementary function ψ by the formula

$$\psi(y) = \sup\{x|y| - \phi(x), x \geq 0\}, \qquad \forall\, y \in \mathbb{R}. \tag{2.2}$$

The complementary function ψ may not be a Young function in the preceding sense. The pair (ϕ, ψ) satisfies the Young inequality

$$xy \leq \phi(x) + \psi(y),\ x \in \mathbb{R},\ y \in \mathbb{R}.$$

Note that equality holds in the Young's inequality if and only if $x = \psi'(y)$ or $y = \phi'(x)$.

Example 2.1.

1. *The complementary functions of* ϕ_p, $1 \leq p < \infty$ *and* $\phi_{\infty,1}$ *defined below are the*

functions ψ_p and $\psi_{\infty,1}$ given by

$$\psi_p(y) = \frac{|y|^q}{q} \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \text{ if } p > 1,$$

$$\psi_1(y) = \begin{cases} 0 \text{ if } -1 \leq y \leq 1, \\ \\ +\infty \text{ if } |y| > 1 \text{ if } p = 1 \end{cases}$$

and

$$\psi_{\infty,1}(y) = \phi_{1,\infty}(y) = \begin{cases} |y| & \text{if } y \in [-1, 1] \\ +\infty & \text{elsewhere.} \end{cases}$$

2. $\phi(x) = e^x - |x| - 1$ *and* $\psi(y) = (1 + |y|)\ln(1 + |y|) - |y|$.

Δ_2-condition

A Young function is said to satisfy the Δ_2-condition (we write $\phi \in \Delta_2$) if there exists $K > 2$ and $u_0 \geq 0$ such that

$$\phi(2u) \leq K\phi(u) \ \forall u \geq u_0 \text{ (see [21, 87]).}$$

Remark 2.1. *When $\phi \notin \Delta_2$ there exists a sequence $(a_n)_{n\geq 1}$ of positive reals numbers increasing to infinity for which*

$$\phi\left(\left(1 + \frac{1}{n}\right)a_n\right) \geq 2^n \phi(a_n), \ \forall n \geq 1 .$$

2.2.2 Orlicz spaces : Definition and examples

Throughout this section I denotes an interval of $\mathbb{R}$ equipped with the Lebesgue measure μ and $M(I)$ the set of all Lebesgue measurable functions defined on the interval I with values in $\mathbb{R}$ modulo the equivalence relation $(= \mu - a.e)$ and $\Sigma(\mathbb{R})$ be the $\sigma-$ algebra of all Lebesgue-measurable subsets of $\mathbb{R}$.

The functional $\rho_\phi : M(I) \to [0, +\infty]$, defined by

$$\rho_\phi(f) = \int_I \phi(|f(t)|)dt$$

is a modular convex on $M(I)$ called Orlicz modular. The associated modular space

$$\begin{aligned} L^{\phi}(I) &= \{f \in M(I),\ \lim_{\lambda \to 0} \rho_{\phi}(\lambda f) = 0\} \\ &= \{f \in M(I),\ \rho_{\phi}(\lambda f) < +\infty \ \ for \ \ some \ \ \lambda > 0\}, \end{aligned}$$

is called Orlicz space (see [21, 61, 87]).

The subspace $E^{\phi}(I)$ of $L^{\phi}(I)$ defined by

$$E^{\phi}(I) = \{f \in M(I),\ \rho_{\phi}(\lambda f) < +\infty \ \ \forall \lambda > 0\}$$

is usually called the subspace of finite elements or "small Orlicz space".
It is well known that $E^{\phi}(I)$ is a closure in $L^{\phi}(I)$ of the set of simple functions with finite measure supports. Moreover, from [21, 61, 82] we know that

$$L^{\phi}(I) = E^{\phi}(I) \ \text{ if and if } \ \phi \in \Delta_2.$$

Remark 2.2. *Note that the Orlicz class $\mathfrak{L}^{\phi}(I)$ defined by*

$$\mathfrak{L}^{\phi}(I) = \left\{ f \in M(I) \ \ such \ that \ \rho_{\phi}(f) < \infty \right\}$$

is not a linear space.

Norms in Orlicz spaces.

Three norms have been established in the theory of Orlicz spaces. In the thirties Orlicz introduced the following norm called **Orlicz norm**.

$$\|f\|_{\phi}^{o} = \sup \left\{ \int_I |f(t)\, g(t)|\, d\mu : g \in L_{\psi}(I), \rho_{\psi}(g)) \leq 1 \right\},$$

where ψ is the complementary of ϕ.
Next, *Nakano* [84] in (1950) , *Morse-Transue* [73] in (1950) and *Luxemburg* in [69] (1955) have considered another norm, which is sometimes called the Luxemburg-Nakano norm but generally in the literature, it is called the Luxemburg norm. This norm is the Minkowski functional of the modular unit ball $(\{f \in L^{\phi}(I),\ \rho_{\phi}(f)) \leq 1\})$.
That is to say,

$$\|f\|_{\phi} = \inf\{\lambda > 0 : \rho_{\phi}\left(\frac{f}{\lambda}\right) \leq 1\}.$$

Around the same time, *I. Amemiya* has considered the norm called "Amemiya norm" defined by

$$\|f\|_{\phi}^{A}=\inf_{k>0}\frac{1}{k}[1+\rho_{\phi}(kf)]\ [21].$$

Note that Orlicz norm is equal to the Amemiya norm (see [50]) and is equivalent to the Luxemburg norm (see [21]). When equipped with one of the above norms, $L^{\phi}(I)$ and $E^{\phi}(I)$ are Banach spaces. However $E^{\phi}(I)$ is always separable, while $L^{\phi}(I)$ is not if ϕ does'nt verify the Δ_2- condition.

Here after, we give some examples of Orilcz spaces equipped with Orlicz norm (see [59]). To simplify notations, we put $L_{\phi}=(L^{\phi},\|.\|_{\phi})$ *and* $L_{\phi}^{0}=(L^{\phi},\ \|.\|_{\phi}^{0})$.

Example 2.2. *[59]*
The Lebesgue spaces L^{p} $(1\leq p<\infty)$ *are Orlicz spaces* $L_{\phi_p}^{o}$ *with* $\phi_p(x)=|x|^{p}$. *Furthermore, if* $(1<p<\infty)$ *we have*

$$\|f\|_{\phi_p}^{o}=\omega_p\|f\|_{L^p}\ \ with\ \ \omega_p=p(p-1)^{\left(\frac{1-p}{p}\right)}.$$

If $p=1$

$$\|f\|_{\phi_1}^{o}=\|f\|_{L^1}.$$

If $p=\infty$ *then,* L^{∞} *is the space* $L_{\phi_\infty}^{o}$ *with* $\phi_\infty:\mathbb{R}\longrightarrow\overline{\mathbb{R}}$ *defined by :*

$$\phi_\infty(x)=\begin{cases}0 & if\ \ x\in[-1,\ 1]\\ +\infty & elsewhere,\end{cases}$$

and

$$\|f\|_{\phi_\infty}^{o}=\|f\|_{L^\infty}.$$

2.2.3 Extreme points of Orlicz spaces

There is a large literature dedicated to the geometry of Orlicz spaces, we can cite for example, Chen's fundamental paper [21] where some of these properties are summarizing.

In this section, we'll go over the results on extreme points of Orlicz spaces. We will consider only the case of Lebesgue measure, for more details about extreme points, strongly extreme points and exposed points in Orlicz spaces equipped with the Luxemburg norm and the Orlicz norm, we refer to [21, 27, 43, 44] and references therein.

Here after, we expose some criteria of special points of the unit ball of Orlicz spaces given in [21].

It is proved in [21] that $x \in \mathbf{extr}[B(L_\phi)]$ if and only if

$$\rho_\phi(x) = 1 \ \ \textit{and} \ \ \mu\left\{t \in I : x(t) \notin S_\phi\right\} = 0.$$

Moreover, criterion that an element in the unit sphere of Orlicz spaces equipped with the Orlicz norm is a extreme point has been obtained, namely $x \in S(L_\phi^0)$ is an extreme point of $B\left(L_\phi^\circ\right)$ if and only if

$$\mu\left\{t \in \Omega : kx(t) \in \mathbb{R}\backslash S_\phi\right\} = 0 \text{ for all } k \in K(x).$$

Note that results on the strict convexity of Orlicz spaces are obtained by several authors, and some of them are obtained by using criteria of extreme points for details we refer to [21].

In [29], Cui and Wang investigated strongly extreme points in Orlicz spaces equipped with the Luxemburg norm. The case of the Orlicz norm is studied by Cui et al. [27].

Theorem 2.1. *$x \in L_\phi$ is an exposed point of $B(L_\phi)$ if and only if*

1. *$\rho_\phi(x) = 1$ and $\mu\{t \in \mathbb{R},\ x(t) \notin S_\phi\} = 0$.*
2. *$p(|x|) \in L_\psi$ and*
3. *$\mu(G(a))\mu(G(b)) = 0$ for all structure affine interval $[a,b]$ of ϕ such that the derivative of ϕ is continuous at a,b.*

   $G(r) = \{t \in \mathbb{R}, |x(t)| = r\}$ and p the right derivative of ϕ.

Theorem 2.2. *$x \in L_\phi^0$ is an exposed point of $B(L_\phi^0)$ if and only if*

1. *$K(x) = \{k\}$ and $\mu\{t \in \mathbb{R} : kx(t) \notin S_\phi\} = 0$.*
2. *For any extreme point γ of any SAI of ϕ, if p is continuous at γ, then $\mu\{t \in G : kx(t) = \gamma\} = 0$.*
3. *If $\rho_\psi(p(k|x|)) = 1$, then $\mu\{t \in G : kx(t) = b\} = 0$ for any SAI $[a,b]$ of ϕ*
4. *If $\theta(kx) < 1$ and $\rho_\psi(p(_{kx(t)})) = 1$, then $\mu\{t \in \mathbb{R} : kx(t) = a\} = 0$ for any Structure affine interval $[a,b]$ of ϕ, where $\theta = \theta(u) = \inf\{\lambda > 0 : \rho_\phi(\frac{u}{\lambda}) < \infty\}$.*

Theorem 2.3. *If $\mathbb{X} = L_{\phi}$ or $X = L^0_{\phi}$ and $x \in S(\mathbb{X})$. Then the following are equivalent :*

1. *x is a denting point of $B(\mathbb{X})$.*
2. *x is an strongly extreme point of $B(\mathbb{X})$.*
3. *x is an extreme point of $B(\mathbb{X})$ and $\phi \in \Delta_2$.*

2.3 Besicovich-Orlicz spaces $B^{\phi}(\mathbb{R},\mathbb{C})$

Let ϕ be a Young function. We denote by $L^{\phi}_{loc}(\mathbb{R},\mathbb{C})$ the subspace of $M(\mathbb{R},\mathbb{C})$ such that for each bounded interval U there exists $\alpha > 0$ such that

$$\int_U \phi\left(\alpha |f(s)|\right) ds < \infty.$$

When $U = [0,1]$, we get *the Orlicz space* $L^{\phi}([0,1],\mathbb{C})$ (see [21]).

The Besicovitch-Orlicz pseudomodular $\rho_{B^{\phi}}$ is defined in [47] as follows

$$\begin{array}{rccl} \rho_{B^{\phi}}: & L^{\phi}_{loc}(\mathbb{R}) & \to & \overline{\mathbb{R}}^{+} \\ & f & \mapsto & \overline{\lim_{T\to+\infty}} \frac{1}{2T} \int_{-T}^{T} \phi|(f(t))|dt. \end{array} \tag{2.3}$$

Its associated modular space, called Besicovitch-Orlicz space, is

$$\mathfrak{B}^{\phi}(\mathbb{R},\mathbb{C}) = \left\{ f \in L^{\phi}_{loc}(\mathbb{R},\mathbb{C}),\ \rho_{B^{\phi}}(\lambda f) < +\infty,\ \text{for some } \lambda > 0 \right\}.$$

This space is endowed with the Luxemburg pseudonorm

$$\|f\|_{B^{\phi}} = \inf\left\{ k > 0, \rho_{B^{\phi}}\left(\frac{f}{k}\right) \leq 1 \right\}.$$

Let us consider the equivalence relation

$$f \sim_{\phi} g \Leftrightarrow \|f-g\|_{B^{\phi}} = 0,\ \forall f,g \in \mathfrak{B}^{\phi}(\mathbb{R},\mathbb{C}).$$

We denote by $B^{\phi}(\mathbb{R},\mathbb{C}) := \mathfrak{B}^{\phi}(\mathbb{R},\mathbb{C}) / \sim_{\phi}$ the quotient space. Henceforth, we will not distinguish between an element of $\mathfrak{B}^{\phi}(\mathbb{R},\mathbb{C})$ and its equivalence class in $B^{\phi}(\mathbb{R},\mathbb{C})$.

Endowed with the Luxemburg norm $\|.\|_{B^{\phi}}$, $B^{\phi}(\mathbb{R},\mathbb{C})$ is a Banach space.

When ϕ is the function $x \mapsto |x|^p$ we obtain the Besicovitch space denoted $B^p(\mathbb{R})$ (see [13]).

New norm in $B^{\phi}(\mathbb{R})$

In 1993 A.G. Aksoy and J.B. Baillon [1] have defined a new norm in Orlicz space as follows :

$$\|f\| = \frac{\xi}{s(f)}$$

where $f \in L_{\phi}$, $\xi > 0$ a fixed real number and $s(f) = \sup\{s : \rho(sf) \leq \xi\} > 0$.
Our objective is to show that the map $f \to \frac{\xi}{\lambda(f)}$ where $f \in B^{\phi}(\mathbb{R})$, $\xi > 0$ a fixed real number and $\lambda(f) = \sup\{s : \rho(sf) \leq \xi\} > 0$ defines a norm in $B^{\phi}(\mathbb{R})$.

Proof. *let $f \in B^{\phi}(\mathbb{R})$, suppose $v > 0$ and consider the new pseudomodular $\rho_v = v\rho_{B^{\phi}}$ then Luxemburg norm $\|f\|^L_{\rho_v}$ we can write*

$$\begin{aligned} \|f\|^L_{\rho_v} &= \inf\{t > 0, \rho_v(\frac{f}{t}) \leq 1\} \\ &= \inf\{t > 0, v\rho_{B^{\phi}}(\frac{f}{t}) \leq 1\} \\ &= \inf\{t > 0, \rho_{B^{\phi}}(\frac{f}{t}) \leq \frac{1}{v}\}. \end{aligned}$$

Take

$$\begin{aligned} i(f) &= \inf\{t > 0, \rho_{B^{\phi}}(\frac{f}{t}) \leq \xi\} \\ &= \inf\{t > 0, \frac{1}{\xi}\rho_{B^{\phi}}(\frac{f}{t}) \leq 1\} = \|f\|^L_{\rho_{\frac{1}{\xi}}}. \end{aligned}$$

Since $\lambda(f) = \frac{1}{i(f)}$, we get

$$\|f\| = \frac{\xi}{\lambda(f)} = \xi i(f) = \xi \|f\|^L_{\rho_{\frac{1}{\xi}}}.$$

Topology and convergence in the Besicovitch-Orlicz space

Let us recall that the modular space $B^{\phi}(\mathbb{R})$ can be with the topology induced by the norm, we consider as open elementary the sets

$$B(f_0, \varepsilon) = \{f \in B^{\phi}(\mathbb{R},\mathbb{C}), \|f - f_0\|_{B^{\phi}} < \varepsilon\}, f_0 \in B^{\phi}(\mathbb{R},\mathbb{C}), \varepsilon > 0.$$

Modular convergence in the modular space $B^{\phi}(\mathbb{R},\mathbb{C})$ is defined as follows

A sequence $(f_n)_{n\geq 1} \subset B^{\phi}(\mathbb{R},\mathbb{C})$ is called modular convergent to $f \in B^{\phi}(\mathbb{R})$, if there is a real number $k > 0$ such that

$$\lim_{n\to+\infty} \rho_{B^{\phi}}(k(f_n - f)) = 0.$$

The following is the relation between modular convergence and convergence in the sense of the norm

$$\lim_{n\to+\infty} \|f_n - f\|_{B^{\phi}} = 0 \text{ if and only if } \lim_{n\to+\infty} \rho_{B^{\phi}}(k(f_n - f)) = 0, \forall k > 0.$$

2.4 Bohr almost periodic functions

The almost periodic functions, initiated by H. Bohr [15], in the mid twenties, are initially defined as a natural generalization of the periodicity in the class of continuous functions. Namely, a continuous function $f : \mathbb{R} \to \mathbb{C}$ is said to be *Bohr almost periodic,* (we write $f \in AP(\mathbb{R},\mathbb{C})$) if for all $\varepsilon > 0$ the set

$$T(f,\varepsilon) := \{\tau \in \mathbb{R},\ \ \|f_\tau - f\|_\infty < \varepsilon\}$$

is relatively dense in $\mathbb{R}$ (see [4], [25]). Recall that the $AP(\mathbb{R},\mathbb{C})$ contains the space $P_w(\mathbb{R},\mathbb{C})$ consisting of all continuous w-periodic functions $f : \mathbb{R} \to \mathbb{C}$ $(w > 0)$. Bohr's theory of almost periodic functions has been extensively studied especially in connections with differential equations.

The main theorem in the theory of the almost periodic functions states that an almost periodic functions can be characterized as uniform limits of sequences from the $Trig(\mathbb{R},\mathbb{C})$: the set of generalized trigonometric polynomials $P_n(t) = \sum\limits_{k=1}^{n} a_k \exp(i\lambda_k t)$, $t \in \mathbb{R}$, where $a_k \in \mathbb{C}$ and $\lambda_1, \lambda_2, ..., \lambda_n$, are mutually different real numbers. More precisely,

$$AP(\mathbb{R},\mathbb{C}) = \overline{Trig(\mathbb{R},\mathbb{C})}^{\|\cdot\|_\infty}.$$

It is this property of almost periodic function which is used at their generalizations : the uniform convergence is replaced by other norms of type L^p or L^{ϕ}.

The previous definition is equivalent to the so-called Bochner's criterion : $f \in AP(\mathbb{R},\mathbb{C})$ if and only if for every sequence of reals $(a_n^{'})$ there exists a subsequence (a_n) such that $(f(\cdot + a_n))$ is uniformly convergent in $BC(\mathbb{R},\mathbb{C})$.

The theory of almost periodic equations has recently been explored in relation to differential equations, stability theory, dynamical systems, and other topics. The theory's variety of applications has grown significantly, now including not just ordinary differential equations and classical dynamical systems, but also a vast variety of partial differential equations and equations Banach spaces.

2.4.1 Properties of Bohr almost periodic functions

Numerical Bohr almost periodic functions have a lot of useful properties, we collect them in the following theorem.

Theorem 2.4. *Let $f \in AP(\mathbb{R},\mathbb{C})$. Then the following holds :*

1. *f is bounded and uniformly continuous.*
2. *If $g \in AP(\mathbb{R},\mathbb{C})$, $h \in AP(\mathbb{R},\mathbb{C})$, $\alpha,\ \beta \in \mathbb{C}$, then $\alpha f + \beta g$ and $hf \in AP(\mathbb{R},\mathbb{C})$;*
3. *$Im(f)$ is relatively compact in $\mathbb{C}$;*
4. *If f' is uniformly continuous then $f' \in AP(\mathbb{R},\mathbb{C})$;*
5. *If $F(t) = \int_0^t f(s)\,ds$ ($t \in \mathbb{R}$) is bounded, then $F \in AP(\mathbb{R},\mathbb{C})$;*
6. *If $(g_n)_{n\in\mathbb{N}}$ is a sequence in $AP(\mathbb{R},\mathbb{C})$ and $(g_n)_{n\in\mathbb{N}}$ converges uniformly to g, then $g \in AP(\mathbb{R}:\mathbb{C})$;*
7. *(Convolution invariance) if $g \in L^1(\mathbb{R})$, then $g * f \in AP(\mathbb{R},\mathbb{C})$, where*

$$(g*f)(t) = \int_{-\infty}^{\infty} g(t-s)f(s)\,ds, \quad t \in \mathbb{R}.$$

8. *The mean value of f*

$$M(f) = \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} f(t)dt = \lim_{T\to+\infty}\frac{1}{2T}\int_{-T+\alpha}^{T+\alpha} f(t)dt,$$

exists and is finite.

9. *Bohr-Fourier coefficient of f,*

$$a(\lambda,f) := \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{-i\lambda s} f(s)\,ds,$$

exists for all $\lambda \in \mathbb{R}$ and

$$a(\lambda,f) := \lim_{T\to\infty}\frac{1}{T}\int_{\alpha}^{T+\alpha} e^{-i\lambda s} f(s)\,ds$$

for all $\alpha \in \mathbb{R}$, $\lambda \in \mathbb{R}$.

10. *If* $a(\lambda, f) = 0$ *for all* $\lambda \in \mathbb{R}$, *then* $f(t) = 0$ *for all* $t \in \mathbb{R}$;
11. *Bohr's spectrum* $\sigma(f) := \{\lambda \in \mathbb{R} : a(\lambda, f) \neq 0\}$ *is at most countable;*
12. *The Perseval equality* $\sum_{n\geq 1} a(\lambda_n, f) = M[f(|x|)^2]$.

2.5 Besicovitch-Orlicz almost periodic functions $B^{\phi}_{a.p}$

2.5.1 Different definitions of $B^{\phi}_{a.p}$

Let us recall that many different definitions are employed in the theory of almost-periodic functions, which are mostly associated to the names of H. Bohr, S. Bochner, V. V. Stepanov, H. Weyl, and A. S. Besicovitch. It is well-known that some of them are equivalent [4, 5, 13], for example definitions of almost periodic functions in terms of a relative density of the set of almost-periods (the Bohr-type criterion), a compactness of the set of translates (the Bochner-type criterion, sometimes called normality), and the closure of the set of trigonometric polynomials in the sup-norm are equivalent (see for example [4, 5]. The same is true for the Stepanov class of almost periodic functions (see [13]). However, if one wants to make some analogy for the Besicovitch class of almost periodic functions, the equivalence is no longer true. Bersani et al. [5] clarify the hierarchy of such classes. In this section we will review Hillmann's definitions of Besicovitch-Orlicz almost periodic functions.

As repeatedly mentioned

$$AP(\mathbb{R}, \mathbb{C}) = \overline{Trig(\mathbb{R}, \mathbb{C})}^{\|.\|_\infty},$$

where $Trig(\mathbb{R}, \mathbb{C})$ is the set of generalized polynomials. The above equality is the starting point for new generalizations of the concept of almost periodicity. Considering the closure of the set $Trig(\mathbb{R}, \mathbb{C})$ with respect to the $\|.\|_{B^\phi}$ T.R. Hillmann [47] has obtained a new classes of almost periodic functions containing the class of Besicovitch of almost periodic functions $B^{p}_{a.p}$ called Besicovitch-Orlicz almost periodic and denoted $B^{\phi}a.p.(\mathbb{R}, \mathbb{C})$.

The general structure as well as certain properties topological of these spaces are studied in [47].

Definition 2.2. *The Besicovitch-Orlicz space of almost periodic functions* $B^{\phi}_{a.p.}$ $resp.(\tilde{B}^{\phi}_{a.p.})$

is the closure of $Trig(\mathbb{R},\mathbb{C})$ in $B^{\phi}(\mathbb{R})$ with respect to the pseudonorm $\|.\|_{B^{\phi}}$ (resp. to the modular convergence), more exactly :

$$B^{\phi}_{a.p.}(\mathbb{R},\mathbb{C}) = \{f \in B^{\phi}(\mathbb{R},\mathbb{C}), \exists (P_n)_{n\geq 1} \subset Trig(\mathbb{R},\mathbb{C}), \ \ s.t \ \lim_{n\to+\infty} \|f-P_n\|_{B^{\phi}} = 0\}$$

$$= \{f \in B^{\phi}(\mathbb{R}), \exists (P_n)_{n\geq 1} \subset Trig(\mathbb{R},\mathbb{C}), \ \ s.t \ \ \forall k>0, \lim_{n\to+\infty} \rho_{B^{\phi}}(k(f-P_n)) = 0\}.$$

$$\tilde{B}^{\phi}_{a.p.}(\mathbb{R},\mathbb{C}) = \{f \in B^{\phi}(\mathbb{R},\mathbb{C}), \exists (P_n)_{n\geq 1} \subset Trig(\mathbb{R},\mathbb{C}), \ \ s.t \ \ \exists k>0, \lim_{n\to+\infty} \rho_{B^{\phi}}(k(f-P_n)) = 0\}.$$

Clearly $B^{\phi}_{a.p.}(\mathbb{R},\mathbb{C}) \subset \tilde{B}^{\phi}_{a.p.}(\mathbb{R},\mathbb{C})$ and equality holds whenever $\phi \in \Delta_2$([47]).

It is known that $B^{\phi}a.p.(\mathbb{R},\mathbb{C}) = \widetilde{B^{\phi}}a.p.(\mathbb{R},\mathbb{C})$ if and only if $\phi \in \Delta_2$ ([79]) From ([79]), we know that when $f \in B^{\phi}_{a.p.}(\mathbb{R})$ the limit exists and is finite in the expression of $\rho_{B^{\phi}}(f)$, i.e.,

$$\rho_{B^{\phi}}(f) = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \phi(|f(t)|)\, d\mu. \tag{2.4}$$

This fact is very useful in our computations.

This space is endowed with the Luxemburg pseudonorm

$$\|f\|_{B^{\phi}} = \inf\left\{k>0, \rho_{B^{\phi}}\left(\frac{f}{k}\right) \leq 1\right\}.$$

Beside the Luxemburg norm Morsli et al [74] have defined in $B^{\phi}_{a.p.}(\mathbb{R})$ another norm, called Orlicz norm, by the formula

$$\|f\|^{o}_{B^{\phi}} = \sup\left\{M(|fg|),\ g \in B^{\psi}_{a.p.}(\mathbb{R}),\ \rho_{B^{\psi}}(g)\ \leq 1\right\}, \tag{2.5}$$

where ψ is the complementary function of ϕ and $M(f) = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{+T} f(t)\, d\mu$.

Remarks 2.1. *If we note by $B^{1}_{a.p}(\mathbb{R},\mathbb{C})$ the space associated with the function $\phi(x) = |x|$, we have*

$$Trig(\mathbb{R},\mathbb{C}) \subset AP(\mathbb{R},\mathbb{C}) \subset B^{\phi}_{a.p.}(\mathbb{R},\mathbb{C}) \subset \tilde{B}^{\phi}_{a.p.}(\mathbb{R},\mathbb{C}) \subset B^{1}_{a.p}(\mathbb{R},\mathbb{C}).$$

The almost periodicity in the sense of Besicovitch-Orlicz was defined in a Bohr-like manner, by using the notion of satisfactorily uniform sets (see e.g. [13] and [5, Definition 5.10, Definition 5.11]).

We'll start with a definition of a numerical set property.

Definition 2.3. *[5]*

A set K is said to be satisfactorily uniform (s.u.) if there exists a positive number l such that the ratio r of the maximum number of elements of K included in an interval of length l to the minimum number is less than 2.

Example : The set $\mathbb{Z}$ is relatively dense and satisfactorily uniform in $\mathbb{R}$.

Remark 2.3. *Every satisfactorily uniform set is relatively dense. In general, the converse is not true. For example, the set $K = \mathbb{Z} \cup \{\frac{1}{n}, n \in \mathbb{N}\}$ is relatively dense, but it is not satisfactorily uniform : this is caused by the presence of the accumulation point 0, $r = +\infty$, $\forall l > 0$. Thus, a relatively dense set, in order to be satisfactorily uniform, cannot have any finite accumulation point.*

Definition 2.4. *([47, 77]) A function $f \in B^{\phi}(\mathbb{R})$ satisfies the $B^{\phi}-$ translation (we write $f \in B^{\phi}_{t.p.}(\mathbb{R})$), if for each $\varepsilon > 0$, there exists a satisfactorily uniform sequence $(\tau_i)_{i\in\mathbb{Z}} \subset \mathbb{R}$ such that :*

1. $\rho_{B^{\phi}}(\frac{f_{\tau_i} - f}{\alpha}) < \varepsilon, \ \ \forall i \in \mathbb{Z}.$
2. $\overline{M}_x \overline{M}_i \{\frac{1}{c}\int_x^{x+c} \phi(\frac{|f(x+\tau_i) - f(x))|}{\alpha}) d\mu\} \leq \varepsilon, \forall i \in \mathbb{Z}, \forall c > 0,$

   where α depends only on the function f.

   The space $B^{\phi}_{t.p.}(\mathbb{R})$ dose not coincide with $B^{\phi}_{a.p.}(\mathbb{R})$ when ϕ dose not obey the Δ_2- condition.

***Approximation property in** $B^{\phi}_{a.p}(\mathbb{R})$ Let f be a function of $B^{\phi}_{a.p}(\mathbb{R})$, we have the following properties [77]*

1. *If P is the Bochner-Fejèr polynomial of f, we have $\rho_{B^{\phi}}(P) \leq \rho_{B^{\phi}}(f)$.*
2. *For all $\varepsilon > 0$, there is a Bochner-Fejèr P_ε polynomial for which we have*

$$\|f - P_\varepsilon\|_{B^{\phi}} \leq \varepsilon.$$

2.5.2 Geometric properties of $B^{\phi}_{a.p}$

We will end this chapter by collecting results obtained on the geometry of $B^{\phi}_{a.p}$

First, M. Morsli [79] has characterized the uniform and strict convexity of the $B^{\phi}_{a.p}$ with Luxemburg norm. Then M. Morsli and F. Bedouhene gave necessary and sufficient *conditions for the strict and uniform convexity of the space* $B^{\phi}_{a.p}$ *equipped with Orlicz norm (see [75, 81].)*

In [8, 76] F. Boulahia and M. Morsli have characterized the uniform non-l^1_n*, the uniform non-squareness and the property* (β) *of* $B^{\phi}_{a.p}$ *in the case of Orlicz norm.*

These properties are summarized in the following table

ϕ *satisfies* $\downarrow$ *if and only if* $\rightarrow$	$(B^{\phi}_{a.p}(\mathbb{R}), \|.\|_{B^{\phi}})$	$(\tilde{B}^{\phi}_{a.p}(\mathbb{R}), \|.\|_{B^{\phi}})$	$(B^{\phi}_{a.p}(\mathbb{R}), \|.\|^0_{B^{\phi}})$
SC			*SC*
SC+Δ_2		*SC*	
SC+UNC.L+Δ_2	*UNC*	*UNC*	
$\Delta_2 \cap \nabla_2$			*UN-NS*
UNC+Δ_2			*UNC+ Pro.*β

CHAPTER 3

Extreme points of the Besicovitch-Orlicz space of almost periodic functions

3.1 Introduction

It is well known that extreme points, which are connected with strict convexity of the space, are the most basic concepts in the geometric theory of Banach spaces see for example [33].

In this chapter we are interested in the extreme points of the Besicovitch-Orlicz spaces of almost periodic functions $B^{\phi}a.p(\mathbb{R},\mathbb{C})$*. As mentioned in the previous chapter, these spaces are the closure of the set of generalized trigonometric polynomials relative to the Luxemburg norm (3), where* ϕ *is a Young or an Orlicz function.*

The general structure as well as certain topological properties of these spaces are studied in [47]. In the recent years, some geometrical properties of $B^{\phi}_{a.p.}(\mathbb{R},\mathbb{C})$ *have been considered by Morsli and his collaborators in [8, 9, 76, 79, 81].*

Morsli [79] has discussed the criteria of rotundity of $B^{\phi}_{a.p.}(\mathbb{R})$ *equipped with Luxemburg norm. He proved that* $B^{\phi}_{a.p.}(\mathbb{R})$ *is strictly convex if and only if* ϕ *is strictly convex and has at most polynomial growth (*ϕ *satisfies the* Δ_2*-condition).*

In [81], Morsli et al have characterized the rotundity of $B^{\phi}_{a.p.}(\mathbb{R})$*, when it is endowed with the Orlicz norm. However, to our knowledge, the criteria for extreme points has not been discussed yet.*

The first objective of this chapter is to characterize the extreme points of the unit ball of $B^{\phi}a.p(\mathbb{R},\mathbb{C})$ equipped with the Luxemburg norm (3), and the second when it is endowed with the Orlicz norm (3.11).

3.2 Extreme points of $B^{\phi}_{a.p}(\mathbb{R},\mathbb{C})$ equipped with the Luxemburg norm

This following section will be devoted to proving some properties of the Besicovitch-Orlicz almost periodic functions.

3.2.1 Properties of the Besicovitch-Orlicz almost periodic functions

We give here some technical results that are helpful to characterize extreme points of $B^{\phi}_{a.p.}(\mathbb{R},\mathbb{C})$ spaces.

First, remember that for a function $f\in B^{\phi}_{a.p.}(\mathbb{R},\mathbb{C})$ we have

$$\rho_{B^{\phi}}(kf)<+\infty,\ \ \forall k>0.$$

Indeed, $f\in B^{\phi}_{a.p.}(\mathbb{R},\mathbb{C})$ then for any $\varepsilon>0$ there exists a trigonometric polynomial P_{ε} such that for any $k>0$

$$\rho_{B^{\phi}}(k(f-P_{\varepsilon}))\leq\frac{\varepsilon}{2}.$$

Then using the convexity of ϕ and the fact that the trigonometric polynomial P_{ε} is bounded we get

$$\rho_{B^{\phi}}(kf)\leq\frac{1}{2}\rho_{B^{\phi}}(2k(f-P_{\varepsilon}))+\frac{1}{2}\rho_{B^{\phi}}(2kP_{\varepsilon})<+\infty.$$

Now, let $\mathscr{P}(\mathbb{R})$ be the family of subsets of $\mathbb{R}$ and $\Sigma(\mathbb{R})$ the $\sigma-$algebra of its Lebesgue measurable sets. Hillmann in [47] has introduced the set function $\overline{\mu}_B:\Sigma(\mathbb{R})\to[0,1]$ as the following

$$\overline{\mu}_B(A)=\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}\chi_A(t)\,d\mu=\overline{\lim_{T\to\infty}}\frac{1}{2T}\mu(A\cap[-T,T]),\tag{3.1}$$

where χ_A denotes the characteristic function of A.

$\overline{\mu}_B$ is not a measure on $\Sigma(\mathbb{R})$: it is increasing, null on sets with $\mu-$finite measure, additive and it is not $\sigma-$additive since

$$\mathbb{R} = \underset{n\in\mathbb{Z}}{\cup}[n,n+1[, \quad \textit{and} \quad \overline{\mu}_B([n,n+1[) = 0.$$

Our first goal in [46] is to show that if $f \in B^{\phi}_{a.p.}(\mathbb{R},\mathbb{C})$ then we do not necessarily have $f\chi_A \in B^{\phi}_{a.p.}(\mathbb{R},\mathbb{C})$ for any $A \in \Sigma(\mathbb{R})$.

Lemma 3.1. *There is a Lebesgue measurable subset A of $\mathbb{R}$, for which the limit*

$$\lim_{T\to\infty}\frac{1}{2T}\mu([-T,T]\cap A), \tag{3.2}$$

does not exist.

Proof. *Let us note first that if the limit (3.2) exists, it would be the same if T is an integer. So to show this lemma, it is sufficient to find a subset $A \in \Sigma(\mathbb{R})$ such that*

$$\lim_{N\to\infty}\frac{1}{N}\mu([0,N]\cap A),$$

does not exist.

Since for $n \geq 1$, there exists $k \in \mathbb{N}$ for which $n \in \left[3^{2k},3^{2k+1}\right[$, we define the sequences $(u_n)_{n\geq 1}$ and $(A_n)_{n\geq 1}$ as following

$$u_n = \begin{cases} 0 & \textit{if} \quad n \in \left[3^{2k},3^{2k+1}\right[\\ 1 & \textit{if} \quad n \in \left[3^{2k+1},3^{2(k+1)}\right[, \end{cases} \tag{3.3}$$

and

$$A_n = \begin{cases} \emptyset & \textit{if} \quad n \in \left[3^{2k},3^{2k+1}\right[\\ [n,n+1[& \textit{if} \quad n \in \left[3^{2k+1},3^{2(k+1)}\right[. \end{cases}$$

We have $\mu(A_n) = u_n$, $\forall n \geq 1$.

Defining $A = \underset{n\geq 1}{\cup}A_n$ and $S_N = \frac{1}{N}\sum\limits_{n=1}^{N}u_n$, we get $\lim\limits_{N\to\infty}\frac{1}{N}\mu\left(A\cap[0,N]\right) = \lim\limits_{N\to\infty}S_N$.

The limit, $\lim\limits_{N\to\infty}S_N$ does not exist. Indeed, for each $N \geq 1$ we have

$$S_{3^{2N}} \quad = \quad \frac{1}{3^{2N}}\sum_{n=1}^{3^{2N}}u_n = \frac{1}{3^{2N}}\sum_{k=0}^{N-1}\left(\sum_{n=3^{2k}}^{3^{2k+1}-1}u_n + \sum_{n=3^{2k+1}}^{3^{2(k+1)}-1}u_n\right) + u_{3^{2N}}$$

$$= \frac{1}{3^{2N}}\sum_{k=0}^{N-1}\left(3^{2(k+1)}-3^{2k+1}\right)$$
$$= \frac{6}{3^{2N}}\sum_{n=1}^{N-1}3^{2k} = \frac{6}{3^{2N}}\left(\frac{3^{2N}-1}{8}\right).$$

It follows that $\lim\limits_{N\to\infty} S_{3^{2N}} = \frac{3}{4}$.

In the other hand,

$$\lim_{N\to\infty} S_{3^{2N+1}} = \lim_{N\to\infty}\left(\frac{1}{3}S_{3^{2N}}+\frac{1}{3^{2N+1}}\right) = \frac{1}{4}.$$

This ends the proof.

Remark 3.1. *If we take f the constant function equal to 1, then using (2.4) and lemma 3.1, we deduce that there exists $A \in \Sigma(\mathbb{R})$ such that $f\chi_A \notin B^{\phi}_{a.p}(\mathbb{R},\mathbb{C})$.*

In the following, we give another property of $\overline{\mu}_B$. We think it can allow us to have characterization more precise for extreme points.

Lemma 3.2. *The function $\overline{\mu}_B : \Sigma(\mathbb{R}) \to [0,1]$ is surjective.*

Proof.

1. *Let $A \in \Sigma(\mathbb{R})$.*
   (a) *It is clearly, by the definition of $\overline{\mu}_B$, that if $\mu(A) < \infty$ we have $\overline{\mu}_B(A) = 0$.*
   (b) *Since $\overline{\mu}_B(\mathbb{R}) = 1$, we obtain $\overline{\mu}_B(A) = 1$ when $\mu(A^c) < \infty$.*
2. *Let $\beta \in]0,1[$, there exists $\alpha > 0$ such that $\beta = \frac{\alpha}{\alpha+1}$. We define*

$$A_n = [(\alpha+1)(n-1),(\alpha+1)(n-1)+\alpha],\ n \in \mathbb{Z}^* \text{ and } A = \bigcup_{n\in\mathbb{Z}^*} A_n.$$

Then,

$$\overline{\mu}_B(A) = \overline{\lim_{T\to+\infty}}\frac{1}{T}\mu\left([0,T]\cap(\bigcup_{n\geq 1}A_n)\right) = \overline{\lim_{T\to+\infty}}\frac{1}{T}\sum_{n\geq 1}\mu(A_n\cap[0,T]).$$

It is easy to show that for any $n \geq 1$,

$$A_n \cap [0,T] = \begin{cases} A_n & \text{if } \; n \leq \lfloor \frac{T+1}{1+\alpha} \rfloor \\ \phi & \text{if } \; n > \lfloor 1 + \frac{T}{\alpha+1} \rfloor \\ \left[\frac{T+1}{1+\alpha}, T\right] & \text{if } \; n \in [\frac{T+1}{1+\alpha}, 1 + \frac{T}{1+\alpha}], \end{cases}$$

where $\lfloor . \rfloor$ denotes the floor function.
Since there is at most one integer in the interval $[\frac{T+1}{1+\alpha}, 1+\frac{T}{1+\alpha}]$, it follows that there exists $0 \leq \theta \leq 1$ such that

$$\begin{aligned} \overline{\mu}_B(A) &= \overline{\lim_{T\to+\infty}} \frac{1}{T} \sum_{n\geq 1} \mu(A_n \cap [0,T]) = \overline{\lim_{T\to+\infty}} \frac{1}{T} \left(\sum_{n=1}^{\lfloor \frac{T+1}{1+\alpha} \rfloor} \mu(A_n) + \theta\alpha \right) \\ &= \overline{\lim_{T\to+\infty}} \frac{1}{T} \left(\lfloor \frac{T+1}{1+\alpha} \rfloor \alpha + \theta\alpha \right) \end{aligned}$$

Using the inequalities : $\forall x \in \mathbb{R},\ x-1 < \lfloor x \rfloor \leq x$, we get $\overline{\mu}_B(A) = \frac{\alpha}{\alpha+1} = \beta$.

In the following we characterize extreme points of the unit ball of $B^{\phi}_{a.p.}(\mathbb{R},\mathbb{C})$. We start with some auxiliary lemmas.

Definition 3.1. A function $f \in B^{\phi}(\mathbb{R},\mathbb{C})$ is said to be absolutely ϕ-integrable in $\bar{\mu}_B$ sense, if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that for every measurable subset $A \in \Sigma(\mathbb{R})$ with $\bar{\mu}_B(A) < \delta$ we have

$$\|f\chi_A\|_{B^\phi} \leq \varepsilon.$$

Lemma 3.3. *Functions $f \in B^{\phi}_{a.p.}(\mathbb{R},\mathbb{C})$ are absolutely ϕ-integrable in the $\bar{\mu}_B$ sense.*

Proof. *First, let us show that bounded functions are absolutely $\phi-$ integrable in $\bar{\mu}_B$ sense.*

Let $\varepsilon > 0$, $A \in \Sigma(\mathbb{R})$ and $f : \mathbb{R} \to \mathbb{C}$ be a bounded function. Put $\mathscr{C} = \sup_{t\in\mathbb{R}} |f(t)|$. Here, we exclude for simplicity the trivial case, when $\bar{\mu}_B(A) = 0$. Clearly, we have

$$\|\chi_A\|_{B^\phi} = \frac{1}{\phi^{-1}(\frac{1}{\bar{\mu}_B(A)})} \;\; \text{and} \;\; \|f\chi_A\|_{B^\phi} \leq \mathscr{C}\,\|\chi_A\|_{B^\phi}.$$

Since the function $t \to \left(\phi^{-1}(1/t)\right)^{-1}$ is continuous and increasing on $]0,+\infty[$, we *deduce that there exists* $\delta := \left(\phi\left(\frac{\mathscr{C}}{\varepsilon}\right)\right)^{-1}$ *such that* $\|f\chi_A\|_{B^\phi} \leq \varepsilon$*, whenever* $\bar{\mu}_B(A) < \delta$.

Now, let us assume that $f \in B^{\phi}_{a.p}(\mathbb{R},\mathbb{C})$*. There exists a trigonometric polynomial* P_ε *such that*

$$\|f - P_\varepsilon\|_{B^\phi} \leq \frac{\varepsilon}{2}. \tag{3.4}$$

Since P_ε *is absolutely* ϕ*-integrable in the* $\bar{\mu}_B$ *sense, there exists* $\delta > 0$ *such that* $\|P_\varepsilon\chi_A\|_{B^\phi} \leq \frac{\varepsilon}{2}$ *whenever* $\bar{\mu}_B(A) < \delta$*. For such* δ*, we have*

$$\|f\chi_A\|_{B^\phi} \leq \|(f - P_\varepsilon)\chi_A\|_{B^\phi} + \|P_\varepsilon\chi_A\|_{B^\phi} \leq \|f - P_\varepsilon\|_{B^\phi} + \|P_\varepsilon\chi_A\|_{B^\phi} \leq \varepsilon.$$

This completes the proof of the lemma.

Let us recall that a sequence $(f_n)_{n\geq 1} \subset B^\phi(\mathbb{R},\mathbb{C})$ *is called*

1. *modular convergent to some* $f \in B^\phi(\mathbb{R},\mathbb{C})$ *when there exists* $\alpha > 0$ *such that*

$$\lim_{n\to\infty} \rho_{B^\phi}(\alpha(f_n - f)) = 0.$$

2. $\overline{\mu}_B$*−convergent to a function* f *when, for all* $\varepsilon > 0$,

$$\lim_{n\to\infty} \overline{\mu}_B\{t \in \mathbb{R},\ |f_n(t) - f(t)| \geq \varepsilon\} = 0.$$

In his work [80], M. Morsli showed that if $(f_n)_{n\in\mathbb{N}}$ *is modular convergent to some* $f \in B^\phi(\mathbb{R},\mathbb{C})$*, it is also* $\overline{\mu}_B$*−convergent to* f*. He also gave in [80] a similar result of the usual Lebesgue dominated convergence theorem in the space* $B^\phi(\mathbb{R},\mathbb{C})$*, as it can be seen in the following proposition.*

Proposition 3.1. *(See [80]) Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence of functions in* $B^\phi(\mathbb{R},\mathbb{C})$*. Then if* $(f_n)_{n\in\mathbb{N}}$ *is* $\overline{\mu}_B$*−convergent to some* $f \in B^\phi(\mathbb{R},\mathbb{C})$ *and there exists* $g \in B^{\phi}_{a.p.}(\mathbb{R},\mathbb{C})$ *such that* $\max(|f_n(x)|, |f(x)|) \leq |g(x)|$, $\forall x \in \mathbb{R}$*. Then,* $\lim_{n\to\infty} \rho_{B^\phi}(f_n) = \rho_{B^\phi}(f)$.

The next Lemmas will appear very useful in the proof of the main result.

Lemma 3.4 (see [9])**.** *Let* $f \in B^{\phi}_{a.p.}(\mathbb{R},\mathbb{C})$ *then*

1. $\|f\|_{B^\phi} \leq 1$ *if and only if* $\rho_{B^\phi}(f) \leq 1$.
2. $\|f\|_{B^\phi} = 1$ *if and only if* $\rho_{B^\phi}(f) = 1$.

Lemma 3.5 (see [79])**.** *Let $f \in B^{\phi}_{a.p.}(\mathbb{R},\mathbb{C})$ such that $\|f\|_{B^{\phi}} = a$, $a > 0$. Then there exists real numbers $0 < \alpha < \beta$ and $\theta \in]0,1[$ such that $\overline{\mu}_B(G) \geq \theta$, where*

$$G = \{t \in \mathbb{R},\ \alpha \leq |f(t)| \leq \beta\}.$$

Lemma 3.6. *Let f be a function in $B^{\phi}_{a.p}(\mathbb{R},\mathbb{C})$, then there exists $\delta > 0$ such that*

$$f\chi_{E^c} \in B^{\phi}_{a.p}(\mathbb{R},\mathbb{C}),$$

for any $E \in \Sigma(\mathbb{R})$ with $\overline{\mu}_B(E) < \delta$. Consequently, $f\chi_E \in B^{\phi}_{a.p}(\mathbb{R},\mathbb{C})$.

Proof. *Let $\varepsilon > 0$, there exists a trigonometric polynomial P_ε such that*

$$\|f - P_\varepsilon\|_{B^{\phi}} \leq \frac{\varepsilon}{2}. \tag{3.5}$$

Using lemma 3.3, there exists $\delta > 0$ such that $\|P_\varepsilon\chi_E\|_{B^{\phi}} \leq \frac{\varepsilon}{2}$, for every measurable subset $E \in \Sigma(\mathbb{R})$ with $\bar{\mu}_B(E) < \delta$.

For the above P_ε, E and δ we have

$$\begin{aligned}\|f\chi_{E^c} - P_\varepsilon\|_{B^{\phi}} &= \|f\chi_{E^c} - P_\varepsilon\chi_{E^c} - P_\varepsilon\chi_E\|_{B^{\phi}} \leq \|(f - P_\varepsilon)\chi_{E^c}\|_{B^{\phi}} + \|P_\varepsilon\chi_E\|_{B^{\phi}} \\ &\leq \|f - P_\varepsilon\|_{B^{\phi}} + \|P_\varepsilon\chi_E\|_{B^{\phi}} \leq \varepsilon.\end{aligned}$$

This show that $f\chi_{E^c} \in B^{\phi}_{a.p}(\mathbb{R},\mathbb{C})$. Hence, the space $B^{\phi}_{a.p}(\mathbb{R},\mathbb{C})$ being linear, we get $f\chi_E \in B^{\phi}_{a.p}(\mathbb{R},\mathbb{C})$.

Lemma 3.7. *[46] Let f be a function in $B^{\phi}_{a.p.}(\mathbb{R})$, then there exists $\delta > 0$ such that*

$$f\chi_{E^c} \in B^{\phi}_{a.p.}(\mathbb{R}),$$

for any $E \in \Sigma(\mathbb{R})$ with $\overline{\mu}_B(E) < \delta$. Consequently, $f\chi_E \in B^{\phi}_{a.p.}(\mathbb{R})$.

Now, using the following Lemma [79, Lemma 4], we give an example of a function in $\widetilde{B}^{\phi}_{a.p.}(\mathbb{R})$.

Lemma 3.8. *Let $(a_n)_{n\geq 1}$, $a_n > 0$ be a sequence of real numbers. With every $n \geq 1$, we associate a measurable set $A_n \subset [0,1[$ such that :*

1. *$A_i \cap A_j = \emptyset$ if $i \neq j$ and $\underset{n\geq 1}{\cup} A_n \subset [0,\alpha[$, where $0 < \alpha < 1$.*

2. $\sum\limits_{n\geq 1}\phi(a_n)\mu(A_n) < +\infty$.
   Consider the function $f = \sum\limits_{n\geq 1} a_n\chi_{A_n}$ *on* $[0,1]$. *Let* $\tilde{f}$ *be the periodic extension of* f *to the whole* $\mathbb{R}$, *with period* $\tau = 1$. *Then* $\tilde{f} \in \tilde{B}^{\phi}_{a.p.}(\mathbb{R})$.

Example 3.1. *Let* $(a_n)_{n\geq 1}$ *and* $(u_n)_{n\geq 1}$ *be two sequences defined by*

$$a_n = \phi^{-1}(\frac{1}{2^n}) \text{ and } u_n = \frac{1}{2^n} \text{ for every } n \geq 1.$$

Put $S_n = \sum\limits_{k=1}^{n} u_k = 1 - \frac{1}{2^n}$. *We define a set sequence* $(A_n)_{n\geq 1}$ *by* $A_n = [S_n, S_{n+1}[$. *Then we have :*

(i) $A_i \cap A_j = \emptyset,\ \forall i \neq j$ *because* $(S_n)_n$ *is strictly increasing.*

(ii) $\lim\limits_{n\to+\infty} S_n = 1$ *which implies that* $\bigcup\limits_{n\geq 1} A_n \subset [0,1[$.

(iii) $\mu(A_n) = u_{n+1},\ \phi(a_n) = \frac{1}{2^n}$ *and* $\sum\limits_{n\geq 1}\phi(a_n)\mu(A_n) = \frac{1}{2}\sum\limits_{n\geq 1}\frac{1}{4^n} < \infty$.

Consider the function defined on $[0,1]$ *by*

$$f = \sum_{n\geq 1}\phi^{-1}(\frac{1}{2^n})\chi_{\left[1-\frac{1}{2^n},1-\frac{1}{2^{n+1}}\right[}.$$

Let $\tilde{f}$ *be the periodic extension of* f *to the whole* $\mathbb{R}$, *with period* $\tau = 1$. *Then by [79, Lemma 4] we have* $\tilde{f} \in \tilde{B}^{\phi}_{a.p.}(\mathbb{R})$.

3.2.2 Extreme points of $B^{\phi}_{a.p}((\mathbb{R},\mathbb{C}), \|.\|_{B^{\phi}})$

We start by recalling some definitions, and auxiliary results we need to prove our main results

Definition 3.2. *A Young function* ϕ *is called*

1. *strictly convex (SC) if*

$$\phi\left(\frac{u+v}{2}\right) < \frac{1}{2}\left(\phi\left(u\right) + \phi\left(v\right)\right),\ \forall u,v \in \mathbb{R},\ u \neq v.$$

2. *uniformly convex for large u (UNCL), if for every* $a \in (0,1)$ *there exists* $\delta(a) \in (0,1)$ *such that*

$$\phi(\frac{u+au}{2}) = (1-\delta(a))\big(\frac{\phi(u)+\phi(au)}{2}\big)\ \forall u, |u| \geq d > 0.$$

3. *uniformly convex (UNC) on $[d,+\infty[$ (see [21]), if for any $\varepsilon>0$ there exists $\delta(\varepsilon)\in\,]0,1[$ such that the inequality*

$$\phi(\frac{u+v}{2})\leq(1-\delta(\varepsilon))\frac{\phi(u)+\phi(v)}{2},$$

holds true for all $u,v\in[d,+\infty[$ satisfying $|u-v|\geq\varepsilon\max(|u|,|v|)$.

Recall that if ϕ is strictly convex then it is uniformly convex on any bounded interval (see [21, Proposition 1.4]). Namely, for any $l>0$ and $\varepsilon>0$, and $[c,d]\subset\,]0,1[$ there exists $\delta>0$ such

$$\phi(\lambda u+(1-\lambda)v)\leq(1-\delta)(\lambda\phi(u)+(1-\lambda)\phi(v)) \tag{3.6}$$

for any $\lambda\in[c,d]$ and all $u,v\in\mathbb{R}$ satisfying $|u|\leq l$, $|v|\leq l$ and $|u-v|\geq\varepsilon$.
Following [21], an interval $[a,b]$ is called a structural affine interval of a Young function ϕ, provided that ϕ is affine on $[a,b]$ and it is not affine on either $[a-\varepsilon,b]$ or $[a,b+\varepsilon]$ for any $\varepsilon>0$.

Let $\{[a_i,b_i]\}_i$ be all the structural affine intervals of ϕ. We denote $S_\phi=\mathbb{R}\backslash\left[\bigcup_i]a_i,b_i[\right]$ the set of strictly convex points of ϕ. Clearly, if $u,v\in\mathbb{R}$, $\alpha\in\,]0,1[$ and $\alpha u+(1-\alpha)v\in S_\phi$, then

$$\phi(\alpha u+(1-\alpha)v)<\alpha\phi(u)+(1-\alpha)\phi(v).$$

Now, we give our principal result.

Theorem 3.1. *Let $f\in S\left(B^{\phi}_{a.p}(\mathbb{R},\mathbb{C})\right)$. We suppose that $\overline{\mu}_B(f^{-1}([a,b]))=0$ for any structural affine interval $[a,b]$ of ϕ. Then*
*$f\in$ **extr** $\left[B\left(B^{\phi}_{a.p}(\mathbb{R},\mathbb{C})\right]\right.$ if and only if $\mu\left(\{t\in\mathbb{R},f(t)\notin S_\phi\}\right)=0$.*

Proof. *The proof is inspired from the Proof of [79, Theorem 1]) and [21, Theorem 2.1].*
Sufficiency : *Suppose that there exists $g,h\in S\left(B^{\phi}_{a.p.}(\mathbb{R},\mathbb{C})\right)$ $(g\neq h)$ such that $2f=g+h$. By lemma 3.4 we have $\rho_{B^\phi}(\frac{g+h}{2})=\rho_{B^\phi}(f)=\rho_{B^\phi}(g)=\rho_{B^\phi}(h)=1$.*

Since $\|g-h\|_{B^\phi}\neq 0$, lemma 3.5 ensures the existence of constants $0<\alpha<\beta$ and $\theta\in\,]0,1[$ for which $\overline{\mu}_B(G_1)>\theta$, where

$$G_1=\{t\in\mathbb{R},\alpha\leq|g(t)-h(t)|\leq\beta\}.$$

Define $M=\phi^{-1}\left(\frac{2}{\overline{\mu}_B(G_1)}\right)$, then $M\leq\phi^{-1}\left(\frac{2}{\theta}\right)$.

Denoting $G_2 = \{t \in \mathbb{R},\ s.t\ |g(t)| \geq M\}$ we will have,

$$1 = \rho_{B^{\phi}}(g) \geq \rho_{B^{\phi}}(g\chi_{G_2}) \geq \phi(M)\,\overline{\mu}_B(G_2) = 2\frac{\overline{\mu}_B(G_2)}{\overline{\mu}_B(G_1)}.$$

Consequently, we have

$$\overline{\mu}_B(G_2) \leq \frac{\overline{\mu}_B(G_1)}{2}. \tag{3.7}$$

Consider now the subset Q of $\mathbb{R}^2$ defined by

$$Q = \left\{(u,v) \in \mathbb{R}^2,\ u,v \in \left[-\left(\phi^{-1}\left(\frac{2}{\theta}\right)+\beta\right),\left(\phi^{-1}\left(\frac{2}{\theta}\right)+\beta\right)\right],\ |u-v| \geq \alpha, \frac{u+v}{2} \in S_{\phi}\right\}.$$

Q is compact. Indeed, we have $Q = Q_1 \cap Q_2$, where

$$Q_1 = \left\{(u,v) \in \mathbb{R}^2, u,v \in \left[-(\phi^{-1}(\frac{2}{\theta})+\beta), \phi^{-1}(\frac{2}{\theta})+\beta\right], |u-v| \geq \alpha\right\},$$

and

$$Q_2 = \left\{(u,v) \in \mathbb{R}^2, \frac{u+v}{2} \in S_{\phi}\right\}.$$

Q_1 is a closed bounded set of $\mathbb{R}^2$ so it is compact, and $Q_2 = g^{-1}(S_{\phi})$, where g denotes the continuous map defined on $\mathbb{R}^2$ by $g(u,v) = \frac{u+v}{2}$.

Then Q_2 is closed because $S_{\phi} = \mathbb{R} \setminus (\bigcup_i]a_i, b_i[)$, so $Q = Q_1 \cap Q_2$ is a closed subset of the compact Q_1.

Consequently, we conclude that Q is compact.

Now, let $F : \mathbb{R}^2 \backslash (0,0) \to \mathbb{R}$ the function defined by

$$F(u,v) = \frac{2\phi\left(\frac{u+v}{2}\right)}{\phi(u)+\phi(v)}.$$

F is continuous and $F(u,v) < 1,\ \forall u,v \in Q$.

Since Q is compact, there exists $0 < \delta < 1$ such that $\sup_{(u,v)\in Q} F(u,v) = 1-\delta$.

So we have

$$\phi\left(\frac{u+v}{2}\right) \leq (1-\delta)\frac{\phi(u)+\phi(v)}{2},\ \ \forall (u,v) \in Q. \tag{3.8}$$

Let us define $G = (G_1 \cap E) \backslash G_2$, where $E = \left\{t \in \mathbb{R},\ s.t\ f(t) \in S_{\phi}\right\}$. It is clear that $\forall t \in G, (g(t), h(t)) \in Q$. We have also $\overline{\mu}_B(G) \geq \frac{\theta}{2}$.

Indeed, since $\mu(E^c)=0$ *we have* $\overline{\mu}_B(G_1\cap E)=\overline{\mu}_B(G_1)$. *Then using (3.7) we get*

$$\overline{\mu}_B(G)=\overline{\mu}_B((G_1\cap E)\backslash G_2)\geq\overline{\mu}_B(G_1\cap E)-\overline{\mu}_B(G_2)\geq\theta-\frac{\theta}{2}=\frac{\theta}{2}. \tag{3.9}$$

We denote $\overline{G}=G\cap[-T,T]$ *and* $\rho_T\left(\frac{g+h}{2}\right)=\frac{1}{2T}\int\limits_{-T}^{T}\phi\left(\frac{|g(t)+h(t)|}{2}\right)d\mu$, *then by (3.8) we obtain*

$$\begin{aligned}\rho_T\left(\frac{g+h}{2}\right) &= \frac{1}{2T}\int\limits_{\overline{G}}\phi\left(\frac{|g(t)+h(t)|}{2}\right)d\mu+\frac{1}{2T}\int\limits_{\overline{G}^c}\phi\left(\frac{|g(t)+h(t)|}{2}\right)d\mu\\ &\leq (1-\delta)\frac{1}{2T}\int\limits_{\overline{G}}\frac{1}{2}\left[\phi\left(|g(t)|\right)+\phi\left(|h(t)|\right)\right]d\mu+\frac{1}{2T}\int\limits_{\overline{G}^c}\frac{1}{2}\left[\phi\left(|g(t)|\right)+\phi\left(|h(t)|\right)\right]d\mu\\ &\leq \frac{1}{2T}\int\limits_{-T}^{T}\frac{1}{2}\left[\phi\left(|g(t)|\right)+\phi\left(|h(t)|\right)\right]d\mu-\delta\frac{1}{2T}\int\limits_{\overline{G}}\frac{1}{2}\left[\phi\left(|g(t)|\right)+\phi\left(|h(t)|\right)\right]d\mu\end{aligned}$$

Since ϕ *is an increasing convex function we have*

$$\frac{1}{2}\left(\phi\left(|g(t)|\right)+\phi\left(|h(t)|\right)\right)\geq\phi\left(\frac{|g(t)|+|h(t)|}{2}\right)\geq\phi\left(\frac{|g(t)-h(t)|}{2}\right)$$

Using the fact that $G\subset G_1$ *we obtain*

$$\frac{1}{2}\left(\rho_T\left(g\right)+\rho_T\left(h\right)\right)-\rho_T\left(\frac{g+h}{2}\right) \quad\geq\quad \delta\phi\left(\frac{\alpha}{2}\right)\frac{\mu\left(\overline{G}\right)}{2T}.$$

Letting $T\to+\infty$, *then by (2.4) and (3.1) we get*

$$\frac{1}{2}\left[\rho_{B^{\phi}}\left(g\right)+\rho_{B^{\phi}}\left(h\right)\right]-\rho_{B^{\phi}}\left(\frac{g+h}{2}\right)\geq\delta\phi\left(\frac{\alpha}{2}\right)\overline{\mu}_B\left(G\right).$$

Consequently, by the inequality (3.9), we deduce that

$$1=\rho_{B^{\phi}}\left(\frac{g+h}{2}\right)\leq\frac{1}{2}\left[\rho_{B^{\phi}}\left(g\right)+\rho_{B^{\phi}}\left(h\right)\right]-\delta\phi\left(\frac{\alpha}{2}\right)\overline{\mu}_B\left(G\right)\leq1-\delta\phi\left(\frac{\alpha}{2}\right)\frac{\theta}{2}.$$

Which is absurd. Thus we showed that f *is an extreme point.*
Necessity : *Suppose that* $\mu\left(\left\{t\in\mathbb{R},\,f\left(t\right)\notin S_{\phi}\right\}\right)>0$.
Let $\varepsilon>0$. *Since* $\mathbb{R}\setminus S_{\phi}$ *is the union of at most countably many open intervals, there*

exists an interval $]a,b[$ *such that for* $\varepsilon > 0$

$$\mu\left(\{t \in \mathbb{R} : f(t) \in]a+\varepsilon, b-\varepsilon[\}\right) > 0,$$

and ϕ *is affine on* $[a,b]$*. That is,*

$$\phi(u) = ku + \beta \ \text{ for } \ u \in [a,b] \ \text{ with } \ k \in \mathbb{R}^+ \ \text{ and } \ \beta \in \mathbb{R}.$$

We divide the set $H = \{t \in \mathbb{R} : f(t) \in]a+\varepsilon, b-\varepsilon[\}$ *into two sets* A *and* B*. Then we define*

$$(g(t),h(t)) = \begin{cases} (f(t), f(t)) & if \ \ t \in \mathbb{R}\backslash(A \cup B) \\ (f(t)-\varepsilon, f(t)+\varepsilon) & if \ \ t \in A \\ (f(t)+\varepsilon, f(t)-\varepsilon) & if \ \ t \in B \end{cases} \tag{3.10}$$

Then $g \neq h$*,* $g+h = 2f$ *and* $g,h \in B^{\phi}_{a.p}(\mathbb{R},\mathbb{C})$*. Indeed, we have* $\overline{\mu}_B(H) = 0$ *because* $H \subset f^{-1}([a,b])$*, then by lemma 3.7 we get*

$$f\chi_{H^c}, (f-\varepsilon)\chi_A, (f+\varepsilon)\chi_B \in B^{\phi}_{a.p}(\mathbb{R},\mathbb{C}).$$

Now, we should show that $\|g\|_{B^\phi} \leq 1$*.*

$$\begin{aligned}\rho_T(g) &= \rho_T\left(f\chi_{(A\cup B)^c}\right) + \rho_T(f\chi_A) + \rho_T(f\chi_B) \\ &= \rho_T\left(f\chi_{(A\cup B)^c}\right) + \frac{1}{2T}\int_{[-T,T]\cap A}(k|f(t)-\varepsilon|+\beta)d\mu + \frac{1}{2T}\int_{[-T,T]\cap B}(k|f(t)+\varepsilon|+\beta)d\mu \\ &\leq \rho_T\left(f\chi_{(A\cup B)^c}\right) + \frac{1}{2T}\int_{[-T,T]\cap A}(k|f(t)|+\beta)+k\varepsilon))d\mu + \frac{1}{2T}\int_{[-T,T]\cap B}(k|f(t)|+\beta)+k\varepsilon))d\mu \\ &\leq \rho_T\left(f\chi_{(A\cup B)^c}\right) + \rho_T\left(f\chi_{(A\cup B)}\right) + k\varepsilon\frac{1}{2T}\mu\left([-T,T]\cap(A\cup B)\right)d\mu.\end{aligned}$$

Letting $T \to +\infty$*, we get*

$$\rho_{B^\phi}(g) \leq \rho_{B^\phi}(f) + k\varepsilon\overline{\mu}_B(H) \leq 1.$$

By applying the hypothesis $\overline{\mu}_B(f^{-1}([a,b])) = 0$ *for any structural affine interval* $[a,b]$ *of* ϕ*, we deduce that* $\overline{\mu}_B(H) = 0$*. Then we get*

$$\rho_{B^\phi}(g) \leq \rho_{B^\phi}(f) = 1.$$

Using same arguments, we obtain $\|h\|_{B^\phi} \leq 1$. Which completes the proof.

The following corollary gives sufficient conditions for the strict convexity of the Besicovitch-Orlicz space of almost periodic functions equipped with the Luxemburg norm. Note that the conditions are the same as those given in [79, Theorem 1] when we consider $B^{\phi}_{a.p.}(\mathbb{R},\mathbb{C})$ instead of $\widetilde{B}^{\phi}_{a.p.}(\mathbb{R},\mathbb{C})$. Recall that

$$\widetilde{B}^{\phi}_{a.p.}(\mathbb{R},\mathbb{C}) = \left\{ f \in B^{\phi}(\mathbb{R},\mathbb{C}), \exists (P_n)_{n\geq 1} \subset Trig(\mathbb{R},\mathbb{C}), \exists k > 0 \text{ s.t. } \lim_{n\to\infty} \rho_{B^\phi}(k(f-P_n)) = 0 \right\}.$$

Corollary 3.1. *If ϕ is strictly convex on $\mathbb{R}$ then $B^{\phi}_{a.p}(\mathbb{R},\mathbb{C})$ is strictly convex.*

Proof. *We know that the hypotheses of strict convexity of ϕ on $\mathbb{R}$ means that $S_\phi = \mathbb{R}$ and then by Theorem 3.1 we get* **extr** $\left[B\left(B^{\phi}_{a.p}(\mathbb{R},\mathbb{C})\right]\right. = S\left(B^{\phi}_{a.p}(\mathbb{R},\mathbb{C})\right)$ *and the claim is proved.*

3.3 Extreme points of $B^{\phi}_{a.p.}(\mathbb{R})$ equipped with the Orlicz norm

The criteria for extreme points and strict convexity in Orlicz spaces and Musielak-Orlicz spaces equipped with the Orlicz norm and Amemiya norm, have been obtained earlier see for instance [21, 35, 36, 90, 96] and references therein.

Beside the Luxemburg norm Morsli et al [74] have defined in $B^{\phi}_{a.p.}(\mathbb{R})$ another norm, called Orlicz norm, by the formula

$$\|f\|^{o}_{B^\phi} = \sup\left\{M(|fg|),\ g \in B^{\psi}_{a.p.}(\mathbb{R}),\ \rho_{B^\psi}(g) \leq 1\right\}, \tag{3.11}$$

where ψ is the complementary function of ϕ and $M(f) = \lim\limits_{T\to+\infty} \frac{1}{2T} \int_{-T}^{+T} f(t)\, d\mu$.

In [81], M. Morsli et al have characterized the rotundity of $B^{\phi}_{a.p.}(\mathbb{R})$, when it is endowed with the Orlicz norm. Others geometric properties of $B^{\phi}_{a.p.}(\mathbb{R})$ in the case of Orlicz norm have been considered by M. Morsli and F. Boulahia [8]. But The extreme points are not yet studied.

In this section, we give results obtained in [17]. We give criteria for the existence of extreme points of the Besicovitch-Orlicz space of almost periodic functions equipped with Orlicz norm, and we study some properties of the set $K(f)$, set of attainable points of the Amemiya norm in this space, defined in Proposition 3.2.

In order to apply the results of the proposition 3.2, ϕ will denote, in this section, an Orlicz function (N−function), i.e., $\phi : \mathbb{R} \to \mathbb{R}^+$ is even, convex, vanishing only at zero and $\lim_{|x|\to+\infty} \phi(x) = +\infty$ and $\lim_{x\to 0} \frac{\phi(x)}{x} = 0$ and $\lim_{x\to+\infty} \frac{\phi(x)}{x} = +\infty$. Note that the complementary function ψ of ϕ is also an Orlicz function when ϕ is.

The norm (3.11) *is not easy to deal with. So Morsli et al [74] expressed it by the Amemiya formula* (3.12) *which is far more convenient to make use.*

Proposition 3.2. (*see [74, 81]*)

Let $f \in B^{\phi}_{a.p.}(\mathbb{R})$, $\|f\|_{B^\phi} \neq 0$. Then ,

1. *the Orlicz norm and the Amemiya norm are equal i.e.,*

$$\|f\|^o_{B^\phi} = \inf_{k>0} \frac{1}{k} \left[1 + \rho_{B^\phi}(kf)\right]. \tag{3.12}$$

   Moreover, there is exists $k_0 \in K(f) = \left\{k > 0, \ \ \|f\|^o_{B^\phi} = \frac{1}{k}\left[1 + \rho_{B^\phi}(kf)\right]\right\}$.

2. $\rho_{B^\phi}(\frac{f}{\|f\|^0_{B^\phi}})) \leq 1$.
3. $\|f\|_{B^\phi} \leq \|f\|^o_{B^\phi} \leq 2\|f\|_{B^\phi}$.

These two norms are equivalent, nevertheless, the corresponding geometric properties between them are different. So their extreme points need not be the same.

Theorem 3.2. *[8]*

1. *If $\phi \in \Delta_2$ then*

$$\overline{k} = \inf\{k \in K(f) : \|f\|^0_{B^\phi} = 1, f \in B^{\phi}_{a.p}(\mathbb{R})\} > 1.$$

2. *If ψ, the conjugate function of ϕ, is of Δ_2- type, then the set*

$$Q = \{k \in K(f), a \leq \|f\|^0_{B^\phi} \leq b, f \in B^{\phi}_{a.p}(\mathbb{R})\}$$

   is bounded for each $0 < a \leq b$.

3.3.1 Some properties of set $K(f)$

Now, we prove some basic properties of the set $K(f)$ when $f \in B^{\phi}_{a.p.}(\mathbb{R})$.

Lemma 3.9. *Let $f \in B^{\phi}_{a.p.}(\mathbb{R})$. Then*

$$K(f) = \{k > 0, \|f\|^0_{B^\phi} = \frac{1}{k}(1+\rho_{B^\phi}(kf)\}$$

is an interval.

Proof. *Let $s,m \in K(f)$ such that $s \neq m$, we will prove that $[s,m] \subseteq K(f)$.*

Let $a \in [0,1]$. By the convexity of ϕ we get,

$$\rho_{B^\phi}((as+(1-a)m)f) \leq a\rho_{B^\phi}(sf)+(1-a)\rho_{B^\phi}(mf).$$

It follows that

$$\begin{aligned} 1+\rho_{B^\phi}((as+(1-a)m)f) &\leq 1+a\rho_{B^\phi}(sf)+(1-a)\rho_{B^\phi}(mf) \\ &= a(1+\rho_{B^\phi}(sf))+(1-a)(1+\rho_{B^\phi}(mf)). \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{as+(1-a)m}[1+\rho_{B^\phi}((as+(1-a)m)f)] &\leq \frac{a(1+a\rho_{B^\phi}(sf))}{as+(1-a)m} \\ &+ \frac{(1-a)(1+\rho_{B^\phi}(mf))}{as+(1-a)m} \\ &= \frac{as}{as+(1-a)m}(\frac{1}{s}(1+\rho_{B^\phi}(sf))) \\ &+ \frac{(1-a)m}{as+(1-a)m}(\frac{1}{m}(1+\rho_{B^\phi}(mf))) \\ &= \frac{as}{as+(1-a)m}\|f\|^0_{B^\phi}+\frac{(1-a)m}{as+(1-a)m}\|f\|^0_{B^\phi} \\ &= \|f\|^0_{B^\phi}. \end{aligned} \tag{3.13}$$

In the other hand, by item 1 of the Proposition 3.2 we have

$$\|f\|^0_{B^\phi} = \inf_{k>0}\frac{1}{k}[1+\rho_{B^\phi}(kf)] \leq \frac{1}{as+(1-a)m}[1+\rho_{B^\phi}((as+(1-a)m)f] \tag{3.14}$$

Combining (3.13) and (3.14) we get,

$$\|f\|^0_{B^\phi} = \frac{1}{as+(1-a)m}(1+\rho_{B^\phi}((as+(1-a)m)f),$$

and so, $as+(1-a)m \in K(f)$ *for all* $a \in [0,1]$. *This means that* $[s,m] \subseteq K(f)$.

Now, let us introduce this notation. For $k > 0$ *;* $f \in B^{\phi}_{a.p.}(\mathbb{R})$, *define the following set :*

$$\overline{S}_\phi(f,k) = \{t \in \mathbb{R}, kf(t) \notin S_\phi\},$$

and $S_\phi(f,k)$ *its complementary.*

Lemma 3.10.

Let $f \in B^{\phi}_{a.p.}(\mathbb{R})$, $\|f\|_{B^\phi} \neq 0$. *We suppose that* $\mu(\overline{S}_\phi(f,k)) = 0$ *for any* $k \in K(f)$. *Then* $K(f)$*consists exactly of one element from* $]0,+\infty[$.

Proof. *Suppose that there exists* $m,s \in K(f)$ *such that* $s < m$.

Then $\|(s-m)f\|^0_{B^\phi} > 0$ *because* $\|f\|^0_{B^\phi} \neq 0$.

By lemma 3.5 there exist reals numbers $0 < \alpha < \beta$, *and* $\theta \in]0,1[$ *such that* $\overline{\mu}_B(G) \geq \theta$ *where*

$$G = \{t \in \mathbb{R}, \alpha \leq |(s-m)f(t)| \leq \beta\}.$$

We know that $\mu(\overline{S}_\phi(f,k)) = 0$, *with* $k \in K(f)$ *this implies that*

$$\overline{\mu}_B(S_\phi(f,k) \cap G) = \overline{\mu}_B(G). \text{ Just write } G = (G \cap S_\phi(f,k)) \cup (G \cap \overline{S}_\phi(f,k)). \tag{3.15}$$

By the convexity of ϕ *and item 1 of the Proposition 3.2 we have*

$$\begin{aligned}
\|f\|^0_{B^\phi} &\leq \frac{2}{s+m}\left[1+\rho_{B^\phi}\left(\frac{s+m}{2}f\right)\right] \\
&\leq \frac{2}{s+m}\left[1+\frac{1}{2}\rho_{B^\phi}(sf)+\frac{1}{2}\rho_{B^\phi}(mf)\right] \\
&\leq \frac{2}{s+m}\left[\frac{1}{2}(1+\rho_{B^\phi}(sf))+\frac{1}{2}(1+\rho_{B^\phi}(mf))\right] \\
&= \frac{2}{s+m}\left[\frac{s}{2}\left(\frac{1}{s}[1+\rho_{B^\phi}(sf)]\right)+\frac{m}{2}\left(\frac{1}{m}[1+\rho_{B^\phi}(mf)]\right)\right] \\
&= \frac{2}{s+m}\left[\frac{s}{2}\|f\|^0_{B^\phi}+\frac{m}{2}\|f\|^0_{B^\phi}\right] = \|f\|^0_{B^\phi}.
\end{aligned}$$

So all the inequalities in the above formulae are, in fact, equalities. Therefore $\frac{s+m}{2} \in K(f)$*, and*

$$\rho_{B^\phi}(\frac{sf+mf}{2}) = \frac{1}{2}\left[\rho_{B^\phi}(sf) + \rho_{B^\phi}(mf)\right]. \tag{3.16}$$

Let $a = \|mf\|^0_{B^\phi}$. *Choose* η *such that* $\min(a\eta, \eta) > 1$. *Put*

$$A = \{t \in \mathbb{R}, |mf(t)| > a\eta\}.$$

Using item 2 *of the Proposition 3.2 we get*

$$a = \|mf\|^0_{B^\phi} \geq \|mf\|_{B^\phi} \geq \|mf\chi_A\|_{B^\phi} \geq a\eta\|\chi_A\|_{B^\phi}.$$

Which implies that $\|\chi_A\|_{B^\phi} \leq \frac{1}{\eta} < 1$*, and then, by Remark* **??**, $\rho_{B^\phi}(\chi_A) \leq \frac{1}{\eta}$.
In view of the following implication (see, e.g. [78, lemma 1])

$$\forall \varepsilon > 0, \exists \delta > 0, \ \forall A \in \Sigma(\mathbb{R}), \ \rho_{B^\phi}(\chi_A) \leq \delta \Rightarrow \overline{\mu}_B(A) < \varepsilon, \tag{3.17}$$

we get that $\overline{\mu}_B(A) < \frac{\theta}{4}$.
Let

$$F_1(u,v) = \frac{2\phi(\frac{u+v}{2})}{\phi(u)+\phi(v)}, \text{ for each } (u,v) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

We have $F_1(u,v) < 1$ *for all* $(u,v) \in Q_1$*, where*

$$Q_1 = \{(u,v) \in \mathbb{R}^2, |u| \leq a\eta, |v| \leq a\eta, |u-v| \geq \alpha, \frac{u+v}{2} \in S_\phi\}.$$

Then using the continuity of F_1 *on the compact set* Q_1*, it follows that there exists* $0 < \delta < 1$ *such that*

$$\sup_{(u,v)\in Q_1} F_1(u,v) = 1 - \delta.$$

More precisely, we have

$$\phi(\frac{u+v}{2}) \leq (1-\delta)\frac{\phi(u)+\phi(v)}{2}, \ \forall (u,v) \in Q_1.$$

Let now $t \in (G \cap S_\phi(f,k)) \setminus A$. *Then* $(sf(t), mf(t)) \in Q_1$.
On the other hand we have

$$\overline{\mu}_B((G \cap S_\phi(f,k)) \setminus A) \geq \overline{\mu}_B(G \cap S_\phi(f,k)) - \overline{\mu}_B(A) \geq \frac{3\theta}{4}.$$

Take $\overline{G}=(G\cap S_{\phi}(f,k))\setminus A$, *it follows that*

$$\begin{aligned}
\frac{1}{2T}\int_{-T}^{T}\phi(\frac{|sf(t)+mf(t)|}{2})dt &= \frac{1}{2T}\int_{[-T,T]\cap\overline{G}}\phi(\frac{|sf(t)+mf(t)|}{2})dt\\
&+ \frac{1}{2T}\int_{[-TT]\cap\overline{G}^{c}}\phi(\frac{|sf(t)(t)+mf(t)|}{2})dt\\
&\le (1-\delta)\frac{1}{2T}\int_{[-T,T]\cap\overline{G}}\frac{\phi(|sf(t)|)+\phi(|mf|)}{2}dt\\
&+ \frac{1}{2T}\int_{[-T,T]\cap\overline{G}^{c}}\frac{\phi(|sf(t)|)+\phi(|mf(t)|)}{2}dt\\
&\le \frac{1}{2T}\int_{-T}^{T}\frac{\phi(|sf(t)|)+\phi(|mf(t)|)}{2}dt\\
&- \delta\frac{1}{2T}\int_{[-T,T]\cap\overline{G}}\frac{\phi(|sf(t)|)+\phi(|mf(t)|)}{2}dt\\
&\le \frac{1}{2}\left[\frac{1}{2T}\int_{-T}^{T}\phi(|sf(t)|dt)+\frac{1}{2T}\int_{-T}^{T}\phi(|mf(t)|)dt\right]\\
&- \delta\frac{1}{2T}\int_{[-T,T]\cap\overline{G}}\phi(\frac{|sf(t)-mf(t)|}{2})dt\\
&\le \frac{1}{2}\left[\frac{1}{2T}\int_{-T}^{T}\phi(|sf(t)|dt)+\frac{1}{2T}\int_{-T}^{T}\phi(|mf(t)|)\right]dt-\delta\phi(\frac{\alpha}{2})\frac{\mu(\overline{G})}{2T}.
\end{aligned}$$

Letting T *tend to infinity we get*

$$\rho_{B^{\phi}}(\frac{sf+mf}{2})\le\frac{1}{2}\left[\rho_{B^{\phi}}(sf)+\rho_{B^{\phi}}(mf)\right]-\delta\phi(\frac{\alpha}{2})\overline{\mu}_{B}(\overline{G}).$$

Then we get,

$$\frac{1}{2}(\rho_{B^{\phi}}(sf)+\rho_{B^{\phi}}(mf))-\rho_{B^{\phi}}(\frac{sf+mf}{2})\ge\delta\phi(\frac{\alpha}{2})\overline{\mu}_{B}(\overline{G})\ge\frac{3\theta}{4}\delta\phi(\frac{\alpha}{2})>0.$$

This contradicts equality (3.20). Then, we have necessarily $s=m$.

Remark 3.2. *It follows from lemma 3.10 that if the function* ϕ *is strictly convex then the set* $K(f)$ *consists of exactly one element.*

3.3.2 Extreme points of $B^{\phi}_{a.p}((\mathbb{R},\mathbb{C}),\|.\|^{o}_{B^{\phi}})$

Now, we characterize the extreme points of the unit ball of $B^{\phi}_{a.p.}(\mathbb{R})$ *equipped with the Orlicz norm.*

Theorem 3.3. *Let $f \in S(B^{\phi}_{a.p.}(\mathbb{R}))$.*

f is an extreme point of $B(B^{\phi}_{a.p.}(\mathbb{R}))$ if and only if $\mu\left(\overline{S}_{\phi}(f,k)\right) = 0$ for any $k \in K(f)$.

Proof. ***Sufficiency*** *: Suppose that $\mu\left(\overline{S}_{\phi}(f,k)\right) = 0$ for any $k \in K(f)$ and $f \notin$ **extr** $\left[B(B^{\phi}_{a.p.}(\mathbb{R}))\right]$. So there exists $g,h \in S(B^{\phi}_{a.p}(\mathbb{R}))$ such that*

$$g \neq h \ \text{ and } f = \frac{g+h}{2}.$$

By Proposition 3.2, we know that $K(g) \neq \emptyset$ and $K(h) \neq \emptyset$. For $k_1 \in K(g)$ and $k_2 \in K(h)$ we have $\|k_1 g - k_2 h\|^0_{B^{\phi}} > 0$.

By lemma 3.5 there exist $\alpha, \beta > 0$ and $\theta \in]0,1[$ such that for the set

$$G = \{t \in \mathbb{R},\, \alpha \leq |k_1 g(t) - k_2 h(t)| \leq \beta\},$$

we have $\overline{\mu}_B(G) > \theta$.

In order to simplify the notation, we put $k = \frac{k_1 k_2}{k_1 + k_2}$. By the convexity of ϕ we get

$$\begin{aligned} 2 &= \|g\|^0_{B^{\phi}} + \|h\|^0_{B^{\phi}} = \frac{1}{k_1}\left[1 + \rho_{B^{\phi}}(k_1 g)\right] + \frac{1}{k_2}\left[1 + \rho_{B^{\phi}}(k_2 h)\right] \\ &= \frac{k_1 + k_2}{k_1 k_2}\left[1 + \frac{k_2}{k_1 + k_2}\rho_{B^{\phi}}(k_1 g) + \frac{k_1}{k_1 + k_2}\rho_{B^{\phi}}(k_2 h)\right] \\ &\geq \frac{1}{k}\left[1 + \rho_{B^{\phi}}(kg + kh)\right] = 2.\frac{1}{2k}\left[1 + \rho_{B^{\phi}}(2kf)\right] \qquad (3.18) \\ &\geq 2\|f\|^0_{B^{\phi}} = 2. \qquad (3.19) \end{aligned}$$

This implies that $2k \in K(f)$ and

$$\rho_{B^{\phi}}(2kf) = \frac{k_2}{k_1 + k_2}\rho_{B^{\phi}}(k_1 g) + \frac{k_1}{k_1 + k_2}\rho_{B^{\phi}}(k_2 h). \qquad (3.20)$$

By hypothesis $\mu\left(\overline{S}_{\phi}(f,2k)\right) = 0$. Then, as in (3.15) we get

$$\overline{\mu}_B(G) = \overline{\mu}_B(G \cap S_{\phi}(f,2k)).$$

Let $\xi > 1$. Define the sets

$$A_1 = \{t \in \mathbb{R}, |g(t)| \geq \xi\};\ \ A_2 = \{t \in \mathbb{R}, |h(t)| \geq \xi\}.$$

Our first claim is that

$$1 = \|g\|^0_{B^\phi} \geq \|g\|_{B^\phi} \geq \|g\chi_{A_1}\|_{B^\phi} \geq \xi \|\chi_{A_1}\|_{B^\phi}.$$

thus $\|\chi_{A_1}\|_{B^\phi} \leq \frac{1}{\xi}$. *Similar computations we get also* $\|\chi_{A_2}\|_{B^\phi} \leq \frac{1}{\xi}$.

Then, by using (3.17)*, we get* $\overline{\mu}_B(A_i) < \frac{\theta}{4}$, $\forall i = 1,2$.

Now, we choose $b = \max(k_1,k_2)$ *and consider the set*

$$Q = \{(u,v) \in \mathbb{R}^2, |u| \leq b\xi + \beta, |v| \leq b\xi + \beta, |u-v| \geq \alpha, \frac{k_2}{k_1+k_2}u + \frac{k_1}{k_1+k_2}v \in S_\phi\},$$

and define the map F *on* $\mathbb{R}^2 \setminus \{(0,0)\}$ *by*

$$F(u,v) = \frac{\phi(\frac{k_2}{k_1+k_2}u + \frac{k_1}{k_1+k_2}v)}{\frac{k_2}{k_1+k_2}\phi(u) + \frac{k_1}{k_1+k_2}\phi(v)}.$$

For all $t \in (S_\phi(f,2k) \cap G) \setminus (A_1 \cup A_2)$*, we have* $(k_1g(t), k_2h(t)) \in Q$*. Then, using same arguments as in the Proof of lemma 3.10, there exists* $0 < \delta < 1$ *such that*

$$\begin{aligned} \phi(\frac{k_1k_2}{k_1+k_2}(g(t)+h(t))) &= \phi\left(\frac{k_2}{k_1+k_2}(k_1g(t)) + \frac{k_1}{k_1+k_2}(k_2h(t))\right) \\ &\leq (1-\delta)\left(\frac{k_2}{k_1+k_2}\phi(k_1g(t)) + \frac{k_1}{k_1+k_2}\phi(k_2h(t))\right). \end{aligned}$$

Denote $\Theta = \frac{k_1+k_2}{k_1.k_2}\left[\frac{k_2}{k_1+k_2}\rho_{B^\phi}(k_1g) + \frac{k_1}{k_1+k_1}\rho_{B^\phi}(k_2h) - \rho_{B^\phi}(2kf)\right]$. *By (3.20), we have* $\Theta = 0$.

On the other hand, if we denote $\overline{G} = [-T,T] \cap \Big((S_\phi(f,k) \cap G) \setminus (A_1 \cup A_1)\Big)$*, we have*

$$\begin{aligned} \Theta &\geq \frac{1}{k_1}\rho_{B^\phi}(k_1g\chi_{\overline{G}}) + \frac{1}{k_2}\rho_{B^\phi}(k_2h\chi_{\overline{G}}) \\ &- \frac{k_1+k_2}{k_1.k_2}\left[(1-\delta)\frac{k_2}{k_1+k_2}\rho_{B^\phi}(k_1g\chi_{\overline{G}}) + \frac{k_1}{k_1+k_2}\rho_{B^\phi}(k_2h\chi_{\overline{G}})\right] \\ &\geq \frac{\delta}{k_1}\rho_{B^\phi}(k_1g\chi_{\overline{G}}) + \frac{\delta}{k_2}\rho_{B^\phi}(k_2h\chi_{\overline{G}}) \\ &\geq \frac{2\delta}{b}\Big(\frac{\rho_{B^\phi}(k_1g\chi_{\overline{G}}) + \rho_{B^\phi}(k_2h\chi_{\overline{G}})}{2}\Big) \\ &\geq \frac{2\delta}{b}\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{\overline{G}}\frac{\phi(k_1|g(t)|) + \phi(k_2|h(t)|)}{2}dt \\ &\geq \frac{2\delta}{b}\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{\overline{G}}\phi(\frac{|k_1g(t) - k_2h(t)|}{2})dt \end{aligned}$$

$$\begin{aligned} &\geq \frac{2\delta}{b}\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{G}\phi\Big(\frac{|k_1 g(t)-k_2 h(t)|}{2}\Big)dt \\ &\geq \frac{2\delta}{b}\phi\Big(\frac{\alpha}{2}\Big)\overline{\mu}_B\left(\overline{G}\right) \geq \frac{\delta}{b}\phi\Big(\frac{\alpha}{2}\Big)\theta > 0. \end{aligned}$$

Contradiction with $\Theta = 0$. Therefor the sufficiency is proved.

Necessity *: we will show that if $f \in$* **extr** $\left[B\left(B^{\phi}_{a.p.}\right)\right]$ *then we have*

$$\mu\left(\overline{S}_{\phi}(f,k)\right) = 0 \;\; for \;\; any \;\; k \in K(f).$$

We assume that there exists $k_0 \in K(f)$ such that $\mu\left(\overline{S}_{\phi}(f,k_0)\right) > 0$

Since $\mathbb{R}\setminus S_{\phi}$ is the union of at most countably many open intervals, there exists an interval $]a,b[$ such that for any $\varepsilon > 0$,

$$\mu\left(\{t \in \mathbb{R}, k_0 f(t) \in\,]a+\varepsilon, b-\varepsilon[\}\right) > 0,$$

and that ϕ is affine on $[a,b]$ i.e., for any $t \in [a,b]$, $\phi(t) = \alpha_1 t + \beta_1$, $\alpha_1 > 0$.

Let $\varepsilon > 0$. Taking

$$H' = \{t \in \mathbb{R},\, k_0 f(t) \in]a+\varepsilon, b-\varepsilon[\}.$$

Then there are two subsets A, B of H' such that $0 < \mu(A) < \infty$, $0 < \mu(B) < \infty$ and $A \cap B = \emptyset$.

Indeed, let $\gamma > 0$, we know that $\mathbb{R} = \underset{n\in\mathbb{Z}}{\cup}[\gamma n, \gamma(n+1)[$. Then $H' = \underset{n\in\mathbb{Z}}{\cup} H'_n$, where $H'_n = H' \cap [\gamma n, \gamma(n+1)[$. We have

$$H'_i \cap H'_j = \emptyset,\ \forall i \neq j,\ \mu(H'_n) \leq \mu\big([\gamma n, \gamma(n+1)\,[\big) = \gamma, \text{ and } \mu(H') = \sum_{n\in\mathbb{Z}} \mu(H'_n).$$

Since $\mu(H') > 0$, two cases may arise

1. *$\mu(H') < \infty$. Then there exists at least two sets H'_1, H'_2 such that*

$$0 < \mu(H'_i) < \gamma,\ \forall i = 1,2,\ (just \;\; choose \; \gamma = \frac{\mu(H')}{2}).$$

2. *$\mu(H') = \infty$. There exists infinitely many sets H'_n such that*

$$0 < \mu(H'_n) < \gamma.$$

Now, we put $H = A \cup B$ we define

$$\begin{cases} g(t) = f(t)\chi_{H^c}(t) + (f(t) - \varepsilon)\chi_A(t) + (f(t) + \varepsilon)\chi_B(t) \\ h(t) = f(t)\chi_{H^c}(t) + (f(t) + \varepsilon)\chi_A(t) + (f(t) - \varepsilon)\chi_B(t) \end{cases}$$

Then $g \neq h$ and $g + h = 2f$.

By lemma 3.7, we get

$$f\chi_{H^c}, (f - \varepsilon)\chi_A, (f + \varepsilon)\chi_B \in B^{\phi}_{a.p.}(\mathbb{R}).$$

Which implies that $g, h \in B^{\phi}_{a.p.}(\mathbb{R})$.

Now, we should show that $\|g\|^0_{B^\phi} \leq 1$.

Let $\rho_T(k_0 g) = \frac{1}{2T}\int_{-T}^{T} \phi(|k_0 g(t)|)dt$. Then, by using the fact that ϕ is affine on H we get

$$\begin{aligned}
\rho_T(k_0 g) &= \rho_T(k_0 f\chi_{H^c}) + \rho_T(k_0 f\chi_A) + \rho_T(k_0 f\chi_B) \\
&= \rho_T(k_0 f\chi_{H^c}) + \frac{1}{2T}\int_{[-T,T]\cap A} \phi\Big(k_0(|f(t) - \varepsilon|)\Big)dt + \frac{1}{2T}\int_{[-T,T]\cap B} \phi\Big(k_0|f(t) + \varepsilon|)\Big)dt \\
&\leq \rho_T(k_0 f\chi_{H^c}) + \frac{1}{2T}\int_{[-T,T]\cap A} \phi\Big(k_0|f(t)| + k_0\varepsilon\Big)dt + \frac{1}{2T}\int_{[-T,T]\cap B} \phi\Big(k_0|f(t)| + k_0\varepsilon\Big)dt \\
&\leq \rho_T(k_0 f\chi_{H^c}) + \frac{1}{2T}\int_{[-T,T]\cap A} \Big(\alpha_1 k_0|f(t)| + \beta_1 + \alpha_1 k_0\varepsilon\Big)dt \\
&+ \frac{1}{2T}\int_{[-T,T]\cap B} \Big(\alpha_1 k_0|f(t)| + k_0\beta_1 + \alpha_1 k_0\varepsilon\Big)dt \\
&\leq \rho_T(k_0 f\chi_{H^c}) + \rho_T(k_0 f\chi_H) + \alpha_1 k_0\varepsilon\frac{1}{2T}\mu(H \cap [-T,T]) \\
&\leq \rho_T(k_0 f) + \alpha_1 k_0\varepsilon\frac{1}{2T}\mu(H \cap [-T,T]).
\end{aligned}$$

Then letting $T \to +\infty$ we get

$$\rho_{B^\phi}(k_0 g) \leq \rho_{B^\phi}(k_0 f) + k_0\varepsilon\overline{\mu}_B(H).$$

Since $\overline{\mu}_B(H) = 0$, we obtain

$$\rho_{B^\phi}(k_0 g) \leq \rho_{B^\phi}(k_0 f), \text{with } k_0 \in K(f).$$

It follows that

$$\|g\|^0_{B^\phi} = \inf_{k>0} \frac{1}{k}\left[1+\rho_{B^\phi}(kg)\right] \le \frac{1}{k_0}\left[1+\rho_{B^\phi}(k_0 g)\right] \le \frac{1}{k_0}\left[1+\rho_{B^\phi}(k_0 f)\right] = 1.$$

Using same arguments, we get $\|h\|^0_{B^\phi} \le 1$.

Finally, we have $g,h \in B\left(B^{\phi}_{a.p.}(\mathbb{R})\right)$ *with* $2f = g+h$, *this shows that* $f \notin \mathbf{extr}\left[B\left(B^{\phi}_{a.p}(\mathbb{R})\right)\right]$. *The proof is complete.*

Remark 3.3. *Criteria for the strict convexity of* $B^{\phi}_{a.p.}(\mathbb{R})$ *endowed with the Orlicz norm is known (see [81, Theorem 4.1]), we can easily deduce this result by our main Theorem 3.3.*

Corollary 3.2. *Let* $f \in S(B^{\phi}_{a.p.}(\mathbb{R}))$.

If the set $K(f)$ *consists of exactly one element* ($K(f) = \{\overline{k}\}$, $\overline{k} > 0$), *and* $\mu\left(\overline{S}_\phi(f,\overline{k})\right) < +\infty$ *then* $\mu\left(\overline{S}_\phi(f,\overline{k})\right) = 0$.

Proof. *Assume that* $0 < \mu\left(\overline{S}_\phi(f,\overline{k})\right) < +\infty$. *Then by Theorem 3.3, we deduce that* $f \notin \mathbf{extr}\left[B(B^{\phi}_{a.p}(\mathbb{R}), \|.\|^0_{B^\phi})\right]$.

It follows that there exist two functions g,h, $g \neq h$ *such that*

$$\|g\|^0_{B^\phi} = \|h\|^0_{B^\phi} = 1 \text{ and } f = \frac{g+h}{2}.$$

Using same notations and arguments as in the proof of the necessity of Theorem 3.3, we get

$$\overline{\mu}_B(G) = \overline{\mu}_B(G \cap S_\phi(f,\overline{k})), \tag{3.21}$$

and $\overline{\mu}_B(A_i) < \frac{\theta}{4}$, $\forall i = 1,2$.

Taking $\Omega = (G \cap S_\phi(f,k))) \backslash (A_1 \cup A_2)$ *then*

$$\begin{aligned} \overline{\mu}_B(\Omega) &\ge \overline{\mu}_B(G \cap S_\phi(f,k)) - \overline{\mu}_B(A_1) - \overline{\mu}_B(A_2) \\ &\ge \overline{\mu}_B(G) - \overline{\mu}_B(A_1) - \overline{\mu}_B(A_2) \ge \frac{\theta}{2}. \end{aligned} \tag{3.22}$$

We have $k_1 \in K(g)$, $k_2 \in K(h)$ *and* $\|g\|^0_{B^\phi} = \|h\|^0_{B^\phi} = 1, f = \frac{g+h}{2}$ *with* $g \neq h$. *By similar computations as in (3.18) we get*

$$1 = \|f\|^0_{B^\phi} = \frac{1}{\frac{2k_1k_2}{k_1+k_2}}\left(1+\rho_{B^\phi}(\frac{2k_1k_2}{k_1+k_2}f\right).$$

Then

$$1-\frac{k_1+k_2}{2k_1k_2}-\frac{k_1+k_2}{2k_1k_2}\rho_{B^{\phi}}(\frac{2k_1k_2}{k_1+k_2}f)=0. \tag{3.23}$$

Put $\rho_T(f)=\frac{1}{2T}\int_{-T}^{T}\phi(|f(t)|)dt, \rho_T(g)=\frac{1}{2T}\int_{-T}^{T}\phi(|g(t)|)dt, \rho_T(h)=\frac{1}{2T}\int_{-T}^{T}\phi(|h(t)|)dt,$ *Then*

$$\begin{aligned}
\rho_T(\frac{2k_1k_2}{k_1+k_2}f) &= \frac{1}{2T}\int_{-T}^{T}\phi\left(\frac{2k_1k_2}{k_1+k_2}|f(t)|\right)dt\\
&= \frac{1}{2T}\int_{-T}^{T}\phi\left(|\frac{k_2}{k_1+k_2}(k_1g(t))+\frac{k_1}{k_1+k_2}(k_2h(t))|\right)dt\\
&= \frac{1}{2T}\int_{[-T,T]\cap\Omega}\phi\left(|\frac{k_2}{k_1+k_2}(k_1g(t))+\frac{k_1}{k_1+k_2}(k_1h(t))|\right)dt\\
&+ \frac{1}{2T}\int_{[-T,T]\cap\Omega^c}\phi\left(|\frac{k_2}{k_1+k_2}(k_1g(t))+\frac{k_1}{k_1+k_2}(k_2h(t))|\right)dt\\
&\leq (1-\delta)\frac{1}{2T}\int_{[-T,T]\cap\Omega}\left[\frac{k_2}{k_1+k_2}\phi(k_1|g(t)|)+\frac{k_1}{k_1+k_2}\phi(k_2|h(t)|)\right]dt\\
&+ \frac{1}{2T}\int_{[-T,T]\cap\Omega^c}\left[\frac{k_2}{k_1+k_2}\phi(k_1|g(t)|)+\frac{k_1}{k_1+k_2}\phi(k_2|h(t)|)\right]dt\\
&\leq \frac{1}{2T}\int_{-T}^{T}\left[\frac{k_2}{k_1+k_2}\phi(k_1|g(t)|)+\frac{k_1}{k_1+k_2}\phi(k_1|h(t)|)\right]dt\\
&- \delta\frac{1}{2T}\int_{[-T,T]\cap\Omega}\left[\frac{k_2}{k_1+k_2}\phi(k_1|g(t)|)+\frac{k_1}{k_1+k_2}\phi(k_2|h(t)|)\right]dt.
\end{aligned}$$

We multiply both sides by $-\frac{k_1+k_2}{2k_1k_2}$ *we get*

$$\begin{aligned}
-\frac{k_1+k_2}{2k_1k_2}\rho_T(\frac{2k_1k_2}{k_1+k_2}f) &\geq \frac{-1}{2}\frac{1}{2T}\int_{-T}^{T}\left[\frac{1}{k_1}\phi(k_1|g(t)|)+\frac{1}{k_2}\phi(k_2|h(t)|)\right]dt\\
&+ \frac{\delta}{2}\frac{1}{2T}\int_{[-T,T]\cap\Omega}\left[\frac{1}{k_1}\phi(k_1|g(t)|)+\frac{1}{k_2}\phi(k_2|h(t)|)\right]dt.
\end{aligned}$$

Since $\frac{1}{k_1}>\frac{1}{b}$ *and* $\frac{1}{k_1}>\frac{1}{b}$, *we obtain*

$$\begin{aligned}
-\frac{k_1+k_2}{2k_1k_2}\rho_T(\frac{2k_1k_2}{k_1+k_2}f) &\geq \frac{-1}{2}(\frac{1}{k_1}\rho_T(k_1g)+\frac{1}{k_2}\rho_T(k_1h))+\frac{\delta}{b}\frac{1}{2T}\int_{[-T,T]\cap\Omega}\phi(\frac{|k_1g(t)-k_2h(t)|}{2})dt\\
&\geq \frac{-1}{2}(\frac{1}{k_1}\rho_T(k_1g)+\frac{1}{k_2}\rho_T(k_2h))+\frac{\delta}{b}\phi(\frac{\alpha}{2})\frac{\mu([-T,T]\cap\Omega)}{2T}
\end{aligned}$$

Letting T tend infinity we have

$$\begin{aligned}
-\frac{k_1+k_2}{2k_1k_2}\rho_{B^\phi}(\frac{2k_1k_2}{k_1+k_2}f) &\geq \frac{-1}{2}(\frac{1}{k_1}\rho_{B^\phi}(k_1g)+\frac{1}{k_2}\rho_{B^\phi}(k_2h))+\frac{\delta}{b}\phi(\frac{\alpha}{2})\overline{\mu}_B(\Omega)\\
&\geq \frac{-1}{2}(\frac{1}{k_1}(1+\rho_{B^\phi}(k_1g))+\frac{1}{k_2}(1+\rho_{B^\phi}(k_2h))-\frac{1}{k_1}-\frac{1}{k_2}+\frac{\delta}{b}\phi(\frac{\alpha}{2})\frac{\theta}{2}\\
&= \frac{-1}{2}(\|g\|^0_{B^\phi})+\|h\|^0_{B^\phi}-\frac{k_1+k_2}{k_1k_2})+\frac{\delta}{b}\phi(\frac{\alpha}{2})\frac{\theta}{2}\\
&= -1+\frac{k_1+k_2}{2k_1k_2}+\frac{\delta}{b}\phi(\frac{\alpha}{2})\frac{\theta}{2}.
\end{aligned}$$

This implies

$$1-\frac{k_1+k_2}{2k_1k_2}-\frac{k_1+k_2}{2k_1k_2}\rho_T(\frac{2k_1k_2}{k_1+k_2}f)\geq\frac{\delta}{b}\phi(\frac{\alpha}{2})\frac{\theta}{2}>0,$$

which contradicts the equality (3.23). This contradiction shows that $\mu\left(\overline{S}_\phi(f,\overline{k})\right)=0$.

The following corollary is an immediate consequence of the previous results.

Corollary 3.3. *Let* $f\in S(B^{\phi}_{a.p}(\mathbb{R}),\|.\|^o_{B^\phi})$ *and* $\mu\left(\overline{S}_\phi(f,k)\right)<\infty$ *for every* $k\in K(f)$*. Then the following statements are equivalent :*

1. *the set* $K(f)$ *consists of exactly one element* $K(f)=\{\overline{k}\},\quad \overline{k}>0$*;*
2. $\mu\left(\overline{S}_\phi(f,\overline{k})\right)=0$*;*
3. $f\in \textbf{\textit{extr}}\left[B(B^{\phi}_{a.p.}(\mathbb{R}),\|.\|^o_{B^\phi})\right]$.

Conclusion and Perspectives

With the knowledge that certain points of the unit sphere of a Banach space (such as extreme points, strongly extreme points, exposed points and denting points) are crucial in the understanding of its geometry, we are interested in this work to the characterization of extreme points of the unit ball of the Besicovitch-Orlicz spaces of almost periodic functions $B^{\phi}a.p.(\mathbb{R},\mathbb{C})$.

Our first results are obtained when $B^{\phi}a.p.(\mathbb{R},\mathbb{C})$ *is equipped with Luxemburg norm. Namely, under specific restriction on the structure affine intervals of* ϕ*, a function* f *of the unit sphere of* $B^{\phi}a.p.(\mathbb{R},\mathbb{C})$ *is an extreme point if and only if* $f(t)$ *is a point of strict convexity of* ϕ *almost for all* $t \in \mathbb{R}$*. To prove this claim, we have showed an important result which states that even if* $f \in B^{\phi}a.p.(\mathbb{R},\mathbb{C})$*,* $f\chi_A$ *is not necessarily almost periodic in the sense of Besicovitch-Orlicz. Thanks to this result "on the extreme points", we refound the sufficient conditions for the strict convexity of* $B^{\phi}a.p.(\mathbb{R},\mathbb{C})$ *in the case of the Luxemburg norm.*

In the case where $B^{\phi}a.p.(\mathbb{R},\mathbb{C})$ *is endowed with the orlicz norm (which is equal to the Amemiya norm), we started by showing some properties of the set* $K(f)$ *(set of elements where the infimum is attained in the Amemiya formula for a function* $f \in B^{\phi}a.p.(\mathbb{R},\mathbb{C})$*). In particular, the set* $K(f)$ *is an interval and under some conditions over* ϕ *this set is a singleton. Then, we showed that a function* f *of the unit sphere of* $B^{\phi}a.p.(\mathbb{R},\mathbb{C})$ *is an extreme point if and only if* $kf(t)$ *is a point of strict convexity of* ϕ*, for any* $k \in K(f)$ *and for almost any* $t \in \mathbb{R}$*.*

In the past few years, several interesting results have been obtained concerning the geometric properties of these spaces, such as uniform and strict convexity, uniform non-l_n^1 *(B-convexity), uniform non-squareness, property* β *etc, the results achieved depend closely on the growth and on the strict (or uniform) convexity of the function* ϕ*. However, there are still a number of unanswered questions in this field of study.*

At the end of this work, the perspectives opened up are for example

1. *Characterization of others special points of the unit ball of $B^{\phi}a.p.$ like strongly extreme points, exposed and denting points.*
2. *Investigation of others geometric properties of $B^{\phi}a.p.$*
3. *The study of the extremal structure of the unit ball of the space of Stepanov-Orlicz almost periodic functions*
4. *The investigation into the geometry of others spaces of generalized almost periodic functions and other functional spaces.*

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